Markovian Projection of Stochastic Processes

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Ecole Doctorale de Paris Centre

Thèse de doctorat

Spécialité : Mathématiques

pour obtenir le grade de

Docteur de l'Université Pierre et Marie Curie

présentée par

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Projection markovienne de processus stochastiques

(Markovian Projection of Stochastic Processes)

Sous la direction de Rama Cont

Soutenue le 28 Mai 2012, devant le jury composé de :

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A ma grand-mère Lalia.

Remerciements

Je tiens à exprimer ma profonde gratitude envers Rama Cont, qui m'a accompagnée dans ma découverte de la recherche. Il m'a proposé un sujet particulièrement riche et stimulant et m'a guidée tout au long de mon parcours avec ses intuitions profondes, sa capacité à faire converger des sujets souvent orthogonaux et son attention à des questions essentielles ou actuelles. Son approche subtilement dosée de la recherche constitue un véritable modèle pour moi. Je tiens également à le remercier pour sa disponibilité à toute heure et ses conseils au cours de ma rédaction, qui m'ont permis d'améliorer la qualité de ce manuscrit. Au delà de son rôle de directeur de thèse, ses qualités humaines m'ont guidée au fil de ces dernières années, et je le remercie pour sa bienveillance et sa patience. Ce fut un honneur d'être son étudiante et je garderai en exemple l'éveil intellectuel, le goût pour la recherche et le recul qu'il a pris soin de me transmettre.

Daniel Lamberton et Peter Friz ont accepté de rapporter sur mes travaux et je tiens à les remercier pour leur relecture attentive ainsi que leurs remarques minutieuses qui m'ont permis d'améliorer mon travail. Je suis reconnaissante envers Peter Friz de m'avoir donné l'occasion de présenter certains de mes résultats au 4th Berlin Workshop on Mathematical Finance for Young Researchers.

Je suis également très honorée de la présence d'Elisa Alos, Nathalie Eisenbaum, Archil Gulisashvili, Ashkan Nikeghbali, Gilles Pagès et Peter Tankov dans mon jury : j'ai un profond respect pour leur expertise pour le calcul stochastique ainsi que leur mission d'en démultiplier les champs d'applications.

Au cours de ces dernières années, j'ai eu l'occasion de collaborer avec des chercheurs tout aussi passionnés les uns que les autres. Je tiens tout d'abord à remercier Marc Yor, dont les inoubliables cours sont à l'origine de mon intérêt pour les probabilités, pour m' avoir accompagnée lors de mes premiers pas dans la recherche et transmis sa passion pour le calcul stochastique, dont la maîtrise constitue un modèle à suivre. Je tiens aussi à remercier Jean Bertoin ainsi que Nathalie Eisenbaum pour les nombreuses discussions que j'ai pu avoir avec eux, qui m'ont guidée dans l' élucidation de mes intrigues stochastiques. Je remercie aussi Jean Jacod, Ashkan Nikeghbali, Christoph Schwab et Josef Teichmann pour l'interêt qu'ils ont porté à mes travaux naissants au cours d'une halte à Zurich. Je garderai un excellent souvenir de mes conversations avec Marco Avellaneda, Bruno Dupire et Nicolas Grandchamp des Raux, dont la connaissance des marchés financiers m'a permis d'affiner mes réflexions. Je tiens aussi à remercier Jorge Zubelli, qui m'a ouvert les portes de l'Instituto Nacional de Matematica Pura e Aplicada (IMPA), où j'ai pu avoir des discussions passionnantes et présenter mes travaux. J'ai grâce à lui eu la chance de pouvoir contribuer à une collaboration, qui m'est chère, entre la France et le Brésil. Je remercie aussi Ioannis Karatzas pour ses questions enrichissantes et pertinentes.

Je tiens aussi à remercier mes amis doctorants ou non, pour leur accompagnement au cours de mon doctorat. Je pense en particulier à Hamed Amini, Vincent Bansaye, Amina Bentata, Marie Bernhart, Karim Bounebache, Paul Bourgade, Reda Chhaibi, Romain Deguest, Lucie Delahaye, David Fournier, Sophie Grezes-Rueff, Adil Hamidou, Ryma Hamidou, Yu Hang Kan, Arseniy Kukanov, Emmanuel Jacob, Franois Jaulin, JinBeom Kim, Cherif Lacene, Adrien de Larrad, Sophie Laruelle, Andreea Minca, Amal Moussa, Nicolas Milicet, Joseph Najnudel, Aurélie Ouss, Hashini Rajapaksha, Abass Sagna, Elisabeth Souny, Stavros Varekoudis, Lakshtithe Wagalath, Gilles Wainrib et Cengbo Zheng.

Je remercie aussi le personnel administratif de Paris 6 et tout particulièrement Corentin Lacombe, Phillipe Macé, Isabelle Mariage, Maria Pochot, Jacques Portes et Josette Saman pour leur énergie et leur disponibilité.

Enfin mes dernières pensées reviennent à ma soeur Rabia, à mes frères ainsi qu'à mes parents, qui m'ont toujours poussée dans l'univers des sciences. Ils ont toute mon affection et ma reconnaissance pour leur soutien, sans mesure, au cours de ces dernières années.

Résumé

Cette thèse porte sur l'étude mathématique du problème de projection Markovienne des processus stochastiques : il s'agit de construire, étant donné un processus aléatoire ξ, un processus de Markov ayant à chaque instant la même distribution que ξ. Cette construction permet ensuite de déployer les outils analytiques disponibles pour l'étude des processus de Markov (équations aux dérivées partielles ou équations integro-différentielles) dans l'étude des lois marginales de ξ, même lorsque ξ n'est pas markovien. D'abord étudié dans un contexte probabiliste, notamment par Gyöngy (1986), ce problème a connu un regain d'intêret motivé par les applications en finance, sous l'impulsion des travaux de B. Dupire.

Une étude systématique des aspects probabilistes est entreprise (construction d'un processus de Markov mimant les lois marginales de ξ) ainsi qu'analytiques (dérivation d'une équation integro-différentielle) de ce problème, étendant les résultats existants au cas de semimartingales discontinues et contribue à éclaircir plusieurs questions mathématiques soulevées dans cette littérature. Ces travaux donnent également une application de ces méthodes, montrant comment elles peuvent servir à réduire la dimension d'un problème à travers l'exemple de l'évaluation des options sur indice en finance.

Le chapitre 1 présente le problème de projection Markovienne et discute son lien avec la construction de processus aléatoires à distributions marginales données. Ce chapitre rappelle également quelques résulats sur les opérateurs intégro-différentiels et les problèmes de martingales associés à ces opérateurs.

Le chapitre 2 est consacré à la construction d'un processus de Markov mimant les lois marginales d'une semimartingale d'Ito ξ, somme d'un processus continu à variation finie, d' une intégrale stochastique Brownienne et de sauts décrits par une mesure aléeatoire M à valeurs entieres:

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int_{\|y\|\leq 1} y\, \tilde{M}(ds\, dy) + \int_0^t \int_{\|y\|>1} y\, M(ds\, dy).$$

Sous l'hypothèse que *soit* ${}^t\delta.\delta$ est uniformément elliptique *soit* ξ est un processus à sauts pur ($\delta = 0$) dont le compensateur des sauts a une singularité de type α-stable en 0, nous construisons un processus de Markov X qui a la même distribution que ξ à chaque instant (Théorème 2.2). Un outil clé est un résultat d'unicité pour l'équation de Kolmogorov forward associée à un opérateur integro-différentiel non-dégénéré (Théorème 2.1). X est défini comme la solution d'un problème de martingale associé à un opérateur integro-différentiel dont les coefficients sont exprimés comme espérance conditionnelle des caractéristiques locales de ξ. Le reste du chapitre consiste à montrer que cette construction s'applique à de nombreux exemples de processus aléatoires rencontrés dans les applications. La construction du Chapitre 2 permet en particulier de montrer que la distribution marginale d'une semimartingale est l'unique solution d'une équation integro-differentielle (EID) "forward": il s'agit de l'équation de Kolmogorov vérifiée par le processus de Markov X qui "mime" ξ.

Le chapitre 3 donne une dérivation analytique directe de ce résultat, sous des hypothèses plus faibles. Ces résultats permettent de généraliser l'équation de Dupire (1995) pour les prix d'options à une large classe de semimartingales discontinues, dont ce chapitre présente plusieurs exemples. L'enoncé des résultats distingue bien les hypothèses sous lesquelles la valeur de l'option vérifie cette EID, des conditions qui garantissent qu'elle en est l'unique solution.

Le chapitre 4 étudie le comportement asymptotique, à temps petit, de quantités de la forme $E[f(X_t)]$ pour des fonctions f régulières, généralisant les résultats existants dans le cas où ξ est un processus de Lévy ou un processus de diffusion au cas général de semimartingales discontinues.

Le chapitre 5 donne une application de ces résultats à l'évaluation d'options sur indice. L'application des résultats du Chapitre 3 permet de réduire ce problème de grande dimension ($d \sim 100$) à la résolution d'une équation integro-différentielle unidimensionnelle, et l'utilisation des résultat du Chapitre 5 permet d'obtenir une approximation numérique dont la complexitée est linéaire (et non exponentielle) avec la dimension. La précision numérique de cette approximation est montrée sur des exemples.

Mots clés : projection markovienne, équation de Kolmogorov, semimartingale, problème de martingale, équation forward, équation de Dupire.

Abstract

This PhD thesis studies various mathematical aspects of problems related to the Markovian projection of stochastic processes, and explores some applications of the results obtained to mathematical finance, in the context of semimartingale models.

Given a stochastic process ξ, modeled as a semimartingale, our aim is to build a Markov process X whose marginal laws are the same as ξ. This construction allows us to use analytical tools such as integro-differential equations to explore or compute quantities involving the marginal laws of ξ, even when ξ is not Markovian.

We present a systematic study of this problem from probabilistic viewpoint and from the analytical viewpoint. On the probabilistic side, given a discontinuous semimartingale we give an explicit construction of a Markov process X which mimics the marginal distributions of ξ, as the solution of a martingale problems for a certain integro-differential operator. This construction extends the approach of Gyöngy to the discontinuous case and applies to a wide range of examples which arise in applications, in particular in mathematical finance. On the analytical side, we show that the flow of marginal distributions of a discontinuous semimartingale is the solution of an integro-differential equation, which extends the Kolmogorov forward equation to a non-Markovian setting. As an application, we derive a forward equation for option prices in a pricing model described by a discontinuous semimartingale. This forward equation generalizes the Dupire equation, originally derived in the case of diffusion models, to the case of a discontinuous semimartingale.

Chapter 2 is devoted to the construction of a Markov process mimicking the marginals laws of an Itô semimartingale ξ_t, decomposed as the sum of a finite variation process, a stochastic integral with respect to a Brownian motion and a stochastic integral with respect to an integer-valued random

measure M.

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int_{\|y\|\leq 1} y\, \tilde{M}(ds\, dy) + \int_0^t \int_{\|y\|>1} y\, M(ds\, dy).$$

If either ${}^t\delta_t\delta_t$ is uniformly elliptic or ξ is a pure jump process such that its compensator has a singularity at 0 of α-stable type, we construct a Markov process X which has the same distribution as ξ at each time (Theorem 2.2). One crucial key is the uniqueness result for the forward Kolmogorov equation associated to an integro-differential operator (Theorem 2.1). X is defined as the solution to the martingale problem associated to an integro-differential operator, whose coefficients are expressed in terms of the local characteristics of ξ. Another part of this Chapter consists in showing that one may apply this construction to a large class of stochastic processes used in applications. The construction of Chapter 2 allows in particular to show that the marginal distribution of ξ is the unique solution of an integro-differential equation, the Kolmogorov equation satisfied by the Markov process X mimicking ξ.

Chapter 3 gives a direct analytical derivation of this result, under weaker assumptions. These results allow to generalize the Dupire's equation for the price of call options to a large class of discontinuous semimartingales, for which we give some examples. We distinguish the assumptions under which the value of the call option satisfies this integro-differential equation, and the conditions implying that it is the unique solution of this equation.

Chapter 4 studies the short-time asymptotic behaviour, of the quantities of the form $\mathbb{E}[f(\xi_t)]$ for functions f regular, generalizing existing results in the case when ξ is a Lévy process or a diffusion to the case of a discontinuous semimartingale.

Chapter 5 gives an application of these results to the evaluation of index options. The application of the results of Chapter 3 allow to reduce the problem of high dimension ($d \sim 30$) to the resolution of a unidimensional integro-differential equation and the techniques developed in Chapter 4 allow to obtain a numerical approximation whose complexity grows linearly with the dimension of the problem. The numerical precision of this approximation is shown via examples.

Keywords: Markovian projection, mimicking theorem, semimartingale, martingale problem, forward equation, Dupire equation.

Contents

Chapter 1

Introduction

1.1 Mimicking the marginal distributions of a stochastic process

The mathematical modeling of stochastic phenomena in various fields such as physics, biology and economics has led to the introduction of stochastic models of increasing complexity, whose dynamics may have history-dependent features. Examples of stochastic processes with such path-dependent dynamics may be found in population dynamics, physics and finance. In most cases, such processes may be represented as *semimartingales* [61, 81], which allow for the use of the tools of Ito calculus.

When we are dealing with Markov processes, a wide range of analytical tools are available for computation, simulation and estimation problems related to stochastic models. But, once we go out of the realm of Markovian models, the complexity sharply increases and tools for computing or simulating quantities related to non-Markovian stochastic processes become scarce and are seldom tractable.

However, in many applications one is solely interested in quantities which depend on a stochastic process $(\xi_t)_{t\geq 0}$ through its *marginal distributions* i.e. the distribution ξ_t for different values of t. A prime example is the *option pricing* problem in mathematical finance, where one is interested in computing quantities of the form $E[f(\xi_t)]$ for various classes of functions $f : \mathbb{R}^d \to \mathbb{R}$.

In these situations, the complexity of the problem can be greatly reduced by considering a simpler model, such as low-dimensional Markov process, with the same marginal distributions. Given a process ξ, a Markov process

X is said to *mimic* ξ on the time interval $[0,T]$, if ξ and X have the same marginal distributions:

$$\forall t \in [0,T], \qquad \xi_t \overset{d}{=} X_t. \tag{1.1}$$

The construction of a Markov process X with the above property is called the "mimicking" problem.

First suggested by Brémaud [21] in the context of queues (under the name of 'first-order equivalence), this problem has been the focus of a considerable literature, using a variety of probabilistic and analytic methods [2, 7, 23, 36, 52, 55, 74, 64, 69, 51]. These results have many applications, in particular related to option pricing problems [34, 36] and the related inverse problem of calibration of pricing models given observed option prices [34, 22, 28, 27, 79].

1.1.1 Stochastic processes with given marginal distributions

The mimicking problem is related to the construction of martingales with a given flow of marginals, which dates back to Kellerer [64]. It is known that the flow of marginal distributions of a martingale is increasing in the convex order i.e. for any *convex* function $\phi : \mathbb{R}^d \mapsto \mathbb{R}$,

$$\forall t \geq s, \qquad \int \phi(y)\, p_t(y)\, dy \geq \int \phi(y)\, p_s(y)\, dy, \tag{1.2}$$

Kellerer [64] shows that, conversely, any family of probability distributions with this property can be realized as the flow of marginal densities of a (sub)martingale:

Theorem 1.1 ([64], p.120)**.** *Let $p_t(y)$, $y \in \mathbb{R}^d$, $t \geq 0$, be a family of marginal densities, with finite first moment, which is increasing in the convex order: for any convex function $\phi : \mathbb{R}^d \mapsto \mathbb{R}$, and any $s < t$,*

$$\forall t \geq s, \qquad \int \phi(y)\, p_t(y)\, dy \geq \int \phi(y)\, p_s(y)\, dy. \tag{1.3}$$

Then there exists a Markov process $(X_t)_{t\geq 0}$ such that the marginal density of X_t is $p_t(y)$. Furthermore, X_t is a martingale.

This approach has been recently extended by Yor and coauthors [7, 55, 74] using a variety of techniques. Madan and Yor [74] give three different constructions of Markov process with a prespecified flow of marginal distributions. The first construction is based on the Azéma–Yor [6] solution to the Skorohod embedding problem. The second one follows the method introduced by Dupire [34] involving continuous martingales. The last approach constructs X as a time-changed Brownian motion.

Such approaches may be applied to the mimicking problem by taking as p_t the flow of marginal distributions of some martingale ξ; the above contructions then yield a Markov process X which mimicks ξ.

These constructions emphasize the lack of uniqueness of the mimicking process and show that the mimicking process may in fact have properties which are quite different from ξ. Even in the case where ξ is a Gaussian martingale, the Azéma–Yor approach yields a mimicking process X which is a discontinuous and time-inhomogeneous Markov process [74]. Considering the special case of gaussian marginals, Albin [2] builds a continuous martingale that has the same univariate marginal distributions as Brownian motion, but that is not a Brownian motion. Another striking example is given by Hamza & Klebaner [52], who construct a family of discontinuous martingales whose marginals match those of a Gaussian Markov process.

Note that these constructions assume the martingale property of ξ. We will now consider an alternative approach which does not rely on the martingale property, and is applicable to a semimartingale ξ.

1.1.2 Markovian projection of a stochastic process

Clearly, given a process ξ, there is not uniqueness (in law or otherwise) of the Markov mimicking process X. However, it is clear that some constructions are more 'natural' than others, as we will try to explain.

First, note that the above constructions [74, 52, 2] take as a starting point the flow of marginal distributions p_t of ξ. In many examples, ξ is defined through its local characteristics (e.g. it is given as a stochastic integral or the solution of a stochastic differential equation) but its marginal distribution may not be known explicitly so these constructions may not be applicable. Indeed, in many cases the goal is to mimic ξ in order to compute p_t!

Also, one would like to view the mimicking process as a 'Markovian projection' of ξ. However, this interpretation fails to hold in the above constructions. In particular, if ξ is already a Markov process then it is natural to choose as mimicking process ξ itself (or at least a copy of ξ with the same law as ξ). However, some of these constructions fail to have this property, so they cannot be qualified as a 'projection' on the set of Markov processes.

Finally, many of these ad-hoc constructions fail to conserve some simple qualitative properties of ξ. For example, if ξ is a continuous process, one expects to be able to mimic it with a continuous Markov process X (preservation of continuity). Also, in reference to the martingale construction above, one can show that it is possible to mimic a (local) martingale ξ with a Markovian (local) martingale X (preservation of the martingale property). These properties are more naturally viewed in terms of the *local characteristics* of the process ξ rather than properties of the marginal distributions. This suggests a more natural, and more general approach for constructing X from the local characteristics of ξ, when ξ is a *semimartingale.*

This approach was first suggested by Krylov [69] and studied in detail by Gyöngy [51] for Ito processes. The following result is given in Gyöngy [51].

Theorem 1.2 ([51], Theorem 4.6)**.** *Consider on a filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$*,* ξ_t *an Ito process*

$$\xi_t = \int_0^t \beta_s(\omega)\, ds + \int_0^t \delta_s(\omega) dW_s,$$

where W *is an* n*-dimensional Wiener process,* δ *and* β *are bounded adapted process taking values in* $M_{d\times n}(\mathbb{R})$ *and* $\mathbb{R}^d$ *respectively. Then under the ellipticity condition*

$${}^t\delta_t.\delta_t \geq \epsilon I_d,$$

there exist bounded measurable functions $\sigma : [0,\infty[\times\mathbb{R}^d \to M_{d\times n}(\mathbb{R})$ *and* $b : [0,\infty[\times\mathbb{R}^d \to \mathbb{R}^d$ *such that*

$${}^t\sigma(t,x).\sigma(t,x) = \mathbb{E}\left[{}^t\delta_t.\delta_t|\xi_t = x\right] \qquad b(t,x) = \mathbb{E}\left[\beta_t|\xi_t = x\right]$$

for almost-every $(t,x) \in [0,\infty[\times\mathbb{R}^d$ *and the stochastic differential equation*

$$dX_t = b(t,X_t)\, dt + \sigma(t,X_t)\, dW_t, \quad X_0 = 0 \tag{1.4}$$

admits a weak solution X_t *which has the same one-dimensional distributions as* ξ_t*.*

The method used by Gyöngy [51] is to construct the Green measure of the process (t,ξ_t) with a given killing rate and identify it with the Green measure of the process (t, X_t) (with another killing rate), where X_t is any solution of the SDE (1.4), starting from 0. The difficulty lies in the fact that σ, b are simply measurable so weak solutions of SDE (1.4) are not unique and X cannot be characterized by an infinitesimal generator. For the same reason, (1.4) cannot be used to construct a Markov process so in this setting one cannot yet speak of a 'Markovian projection'.

Brunick & Shreve [23] extend this result by relaxing the ellipticity condition of [51] but using a totally different approach where the weak solution of the SDE (1.4) is constructed as a limit of mixtures of Wiener measures, following a method suggested by Brigo and Mercurio [22]. As in Gyöngy [51], the mimicking process X is constructed as a weak solution to the SDE (1.4), but this weak solution does *not* in general have the Markov property: indeed, it need not even be unique under the assumptions used in [51, 23] which allow for discontinuity of coefficients. In particular, in the setting used in [51, 23], the law of X is not uniquely determined by its 'infinitesimal generator' defined as

$$\forall f \in C_0^\infty(\mathbb{R}^d) \quad \mathcal{L}_t f(x) = b(t,x).\nabla f(x) + \frac{1}{2}\left[a(t,x)\,\nabla^2 f(x)\right], \tag{1.5}$$

The 'computation' of quantities involving X, either through simulation or by solving a partial differential equation may not be trivial because of this non-uniqueness: for example, one cannot even simulate the solution of the stochastic differential equation under these conditions.

However, what this construction does suggest is that the 'natural' Markovian projection of ξ is the Markov process –if it exists– with infinitesimal generator (1.5). If this Markov process exists and enjoys minimal regularity properties, then one can compute expectations of the form $E[f(\xi_t)|\mathcal{F}_0]$ by solving partial differential equation associated with the operator (1.5).

1.1.3 Forward equations for option prices

These "mimicking theorems" have a natural connection to another strand of literature : mathematical finance, precisely focuses on the derivation of such 'forward equations' for expectations of the type

$$C(T,K) = e^{-r(T-t)}\,E[(\xi_t - K)_+|\mathcal{F}_0],$$

which correspond to the value of call options written on ξ. This connection comes from the fact that call options values are related to the marginal distributions $(p_T)_{T\geq 0}$ of ξ by

$$\frac{\partial^2 C}{\partial K^2}(T,K) = e^{-r(T-t)}\, p_T(dK),$$

so a process which mimics the marginal distributions of ξ will also "calibrate" the prices of call options and forward Kolmogorov equations for p_T translate into "forward equations" for $C(T,K)$. Forward equations for option pricing were first derived by Dupire [34, 36] who showed that when the underlying asset is assumed to follow a diffusion process

$$dS_t = S_t \sigma(t, S_t) dW_t$$

prices of call options (at a given date t_0) solve a *forward* PDE in the strike and maturity variables

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\,\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K)$$

on $[t_0,\infty[\times]0,\infty[$, with the initial condition

$$\forall K>0 \quad C_{t_0}(t_0,K) = (S_{t_0}-K)_+.$$

This forward equation allows to price call options with various strikes and maturities on the same underlying asset, by solving a *single* partial differential equation. Dupire's forward equation also provides useful insights into the *inverse problem* of calibrating diffusion models to observed call and put option prices [16]. As noted by Dupire [35], the forward PDE holds in a more general context than the backward PDE: even if the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale

$$dS_t = S_t\delta_t dW_t,$$

then call options still verify a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function $\sigma(t,S)$ given by

$$\sigma(t,S) = \sqrt{E[\delta_t^2|S_t = S]}.$$

Given the theoretical and computational usefulness of the forward equation, there have been various attempts to extend Dupire's forward equation to other types of options and processes, most notably to Markov processes with jumps [4, 26, 29, 62, 25]. Most of these constructions use the Markov property of the underlying process in a crucial way.

A common mistake is to state that Dupire's forward PDE is a consequence of Gyöngy [51] mimicking theorem. As pointed out in the previous section, the construction of Gyöngy's [51] does *not* imply a link with the forward PDE and the derivations of forward PDEs do *not* make use of Gyöngy's mimicking result or any other construction of the mimicking process, but directly derive a PDE or PIDE for the function $C(T,K) = e^{-r(T-t)}\, E[(\xi_t - K)_+|\mathcal{F}_0]$. The reason Gyöngy's result does not allow the derivation of such a PDE is that the underlying assumptions do not give sufficient regularity needed to derive the PDE for $C(T,K)$.

However, intuitively there *should* be a link between these two strands of results. It therefore remains to describe the link between these two approaches: on the one hand, the construction of the mimicking process –or Markovian projection– of ξ and, on the other hand, the derivation of forward equations for $C(T,K) = E[(\xi_t - K)_+|\mathcal{F}_0]$. This is precisely one of the objectives of this thesis. In order to explore this relation, we need to construct, not just a mimicking process which is 'Markov-like' as in [51, 23] but a *Markovian projection* X which is a genuine Markov process, characterized by its infinitesimal generator and whose marginal distributions may be characterized as the unique solution of the corresponding Kolmogorov forward equation. Then, we will be able to connect the dots and establish a one-to-one correspondence between the mimicking process, the forward equation for call options and the infinitesimal generator of X.

We will show that this construction is indeed possible when ξ is an *Ito semimartingale*:

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int_{\|y\|\leq 1} y\, \tilde{M}(ds\, dy) + \int_0^t \int_{\|y\|>1} y\, M(ds\, dy), \tag{1.6}$$

where ξ_0 is in $\mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, M is an integer-valued random measure on $[0,\infty) \times \mathbb{R}^d$ with compensator measure μ and $\tilde{M} = M - \mu$ is the compensated measure associated to M. Namely, μ is the unique, up to a $\mathbb{P}$-null set, predictable random measure on $[0,\infty[\times\mathbb{R}^d$ such that

$$\int_0^{\cdot} \int_{\mathbb{R}^d} \phi_t\, (M(dt\, dy) - \mu(dt\, dy)) \text{ is a } (\mathbb{P},\, \mathcal{F}_t) - \text{local martingale}, \tag{1.7}$$

(see [61, Ch.II,Sec.1]), implying that for any predictable process ϕ_t and for any $T > 0$:

$$\mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \phi_t M(dt\, dy)\right] = \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} \phi_t \mu(dt\, dy)\right]. \tag{1.8}$$

Ito semimartingales form a natural class of stochastic processes, sufficiently large for most applications and possessing various analytical properties which allow in particular the use of stochastic calculus [81, 61].

An Ito semimartingale may be characterized by its *local characteristic triplet* (β, δ, μ), which may be seen as a path-dependent generalization of the notion of Lévy triplet for Lévy processes.[1] Under some conditions on the *local characteristics* of ξ, we will show that the Markovian projection X of ξ may then be constructed, as in Gyöngy [51], by projecting the local characteristics of ξ on its state. However, our construction differs from that of Gyöngy: we construct X as the solution of a *martingale problem* and ensure that this construction yields a Markov process X, characterized by its infinitesimal generator. This regularity of the construction will provide a clear link between the mimicking process X and the corresponding forward equation and clarify the link between forward equations and Markovian projections.

1.1.4 Stochastic differential equations and martingale problems

To construct Markovian mimicking processes for ξ, we will need to construct solutions to a general 'Markovian-type' stochastic differential equation with jumps, given by

$$\begin{aligned}\forall t \in [0,T], \qquad X_t = X_0 &+ \int_0^t b(u, X_u)\, du + \int_0^t \Sigma(u, X_u)\, dB_u \\ &+ \int_0^t \int_{\|y\|\le 1} y\, \tilde{N}(du\ dy) + \int_0^t \int_{\|y\|>1} y\, N(du\ dy),\end{aligned} \tag{1.9}$$

where (B_t) is a d-dimensional Brownian motion, N is an integer-valued random measure on $[0,T] \times \mathbb{R}^d$ with compensator $n(t, dy, X_{t-})\, dt$ where $(n(t, \,.\,, x), (t,x) \in [0,T] \times \mathbb{R}^d)$ is a measurable family of positive measures

[1] We refer to Jacod & Shiryaev [61, Chapter 4 Section 2] for a complete presentation.

on $\mathbb{R}^d - \{0\}$, $\tilde{N} = N - n$ the associated compensated random measure, $\Sigma \in C^0([0,T]\times\mathbb{R}^d, M_{d\times n}(\mathbb{R}))$, $b : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ are measurable functions.

When the coefficients (b, Σ, n) are Lipschitz with linear growth, this equation has a unique strong (pathwise) solution which may be constructed by Picard iteration [58]. However, such regularity properties are in general not available in the examples we shall examine. Furthermore, since in the mimicking problem we are only interested in the distributional properties of processes, it is sufficient to construct the Markov mimicking process X as a "weak" –or probabilistic– solution of (1.9). This can be achieved under much weaker regularity conditions on the coefficients. A systematic construction of such weak solutions was proposed by Stroock and Varadhan [90], who introduced the notion of martingale problem associated to an operator L verifying the maximum principle.

If X is a Markov process on a stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and L denotes its infinitesimal generator, defined on a given domain $\mathcal{D}(L)$, then for $f \in \mathcal{D}(L)$,

$$M(f)_t = f(X_t) - f(X_0) - \int_0^t Lf(X_s)\,ds \tag{1.10}$$

is a $\mathbb{P}$-martingale. This property, which characterizes the law of X in terms of its *infinitesimal generator* L, was used by Stroock and Varadhan [90] as a systematic method for constructing weak solutions of stochastic differential equations as solutions to "martingale problems" [90].

Let $\Omega_0 = D([0,T],\mathbb{R}^d)$ be the Skorokhod space of right-continuous functions with left limits, denote by $X_t(\omega) = \omega(t)$ the canonical process on Ω_0, $\mathcal{B}_t^0$ its filtration and $\mathcal{B}_t$ the $\mathbb{P}-$completed right-continuous version of $\mathcal{B}_t^0$. Let $\mathcal{C}_b^0(\mathbb{R}^d)$ denote the set of bounded and continuous functions on $\mathbb{R}^d$ and $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the set of infinitely differentiable functions with compact support on $\mathbb{R}^d$. Consider a time-dependent integro-differential operator $\mathcal{L} = (\mathcal{L}_t)_{t\in[0,T]}$ defined, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, by

$$\begin{aligned}\mathcal{L}_t f(x) = b(t,x).\nabla f(x) + \frac{1}{2}\mathrm{tr}\left[a(t,x)\,\nabla^2 f(x)\right] \\ + \int_{\mathbb{R}^d} [f(x+y) - f(x) - 1_{\{\|y\|\leq 1\}}\, y.\nabla f(x)] n(t,dy,x),\end{aligned} \tag{1.11}$$

where $a : [0,T]\times\mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R}), b : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ are measurable functions and $(n(t,\,.\,,x), (t,x) \in [0,T]\times\mathbb{R}^d)$ is a measurable family of positive measures on $\mathbb{R}^d - \{0\}$.

For x_0 in $\mathbb{R}^d$, a probability measure $\mathbb{Q}_{x_0}$ on $(\Omega_0, \mathcal{B}_T)$ is said to be a solution to the *martingale problem* for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ if $\mathbb{Q}\,(X_0 = x_0) = 1$ and for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the process

$$f(X_t) - f(x_0) - \int_0^t \mathcal{L}_s f(X_s)\, ds$$

is a $(\mathbb{Q}_{x_0}, (\mathcal{B}_t)_{t\geq 0})$-martingale on $[0,T]$.

If for any $x_0 \in \mathbb{R}^d$, there exists a unique solution $\mathbb{Q}_{x_0}$ to the martingale problem for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$, then we say that the martingale problem for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ is well-posed.

Ethier & Kurtz [40, Chapter 4, Theorem 4.2] show that the well-posedness of the martingale problem for $(L, \mathcal{D}(L)$ implies the Markov property of X under $\mathbb{Q}$. Hence when one wants to build a Markovian semimartingale whose infinitesimal generator is an integro-differential operator such as $\mathcal{L}$, the existence and uniqueness of solutions to martingale problems for integro-differential operators are crucial.

Existence and uniqueness for integro-differential operator have been studied under various conditions on the coefficients. When $\mathcal{L}$ has constant coefficients $b(t,x) = b$, $a(t,x) = a$ and $n(t,dy,x) = n(dy)$, then $\mathcal{L}$ is the infinitesimal generator of a pure jump Lévy process (see Bertoin [19], Sato [84]). The existence of a solution to the martingale problem for the operator $\mathcal{L}$ with continuous or measurable coefficients and nondegenerate diffusion matrix has been considered by Skorokhod [87], Stroock and Varadhan [90], Krylov [68], Lepeltier and Marchal [72] and Jacod [59]. The uniqueness for a continuous nondegenerate diffusion coefficient was studied by Stroock and Varadhan [90], Komatsu [66] and Stroock [88]. In all these results, boundedness of coefficients guarantees existence, while uniqueness is based on some form of continuity of coefficients plus an ellipticity condition of the diffusion matrix. Figalli [42] extends these results to the case of second-order differential operators with irregular coefficients.

Komatsu [67] was the first to treat the martingale problem for pure jump processes generated by operators where $a = 0$. In the case where b and n are non-time dependent Komatsu [67] proved the existence and uniqueness of solutions for Lévy Kernels $n(dy,x)$ which are perturbations of an α-stable Lévy measure. Uniqueness in the case of a state dependent singularity at zero was proved by Bass [10, 11] under some uniform continuity the intensity of small jumps with respect to the jump size.

Mikulevicus and Pragarauskas [77] improved the results of Stroock [88] and Komatsu [66], [67], by allowing a, b and n to be time dependent. They use a different approach than Stroock [88] or Komatsu [67] by estimating the unique solution to the Cauchy problem for nondegenerate Lévy operators in Sobolev and Hölder spaces. More recently, using tools from complex analysis and pseudo-differential operators, Hoh [56, 57] explored the well-posedness of the martingale problem for a non-time dependent operator, when $x \to n(dy, x)$ is in $\mathcal{C}^{3d}(\mathbb{R}^d)$.

We will mainly use the results of Mikulevicus and Pragarauskas [77] which cover the case of time-dependent coefficients and whose conditions are easier to verify in applications.

Lepeltier [72] shows that the notion of solution to martingale problem is equivalent to the notion of weak solution for a stochastic differential equation with jumps. [72, Theorem II_9] shows that if X is the weak solution of the stochastic differential equation (1.9), then if one defines

$$a(t,x) = {}^t\Sigma(t,x)\Sigma(t,x),$$

the probability measure $\mathbb{P}_{X_0}$ is a solution the *martingale problem* for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ defined by equation (1.11) with initial condition X_0. Conversely, [72, Theorem II_{10}] shows that if $\mathbb{P}_{X_0}$ is a solution to the *martingale problem* for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ starting from X_0 then, for any $\Sigma \in C^0([0,T] \times \mathbb{R}^d, M_{d\times n}(\mathbb{R}))$ such that

$$ {}^t\Sigma(t,x)\Sigma(t,x) = a(t,x) \tag{1.12}$$

$\mathbb{P}_{X_0}$ is a weak solution to the stochastic integro-differential equation (1.9) with initial condition X_0.

Furthermore, [72, Corollary II_{10}] shows that the uniqueness in law of the stochastic integro-differential equation (1.9) for a given X_0 implies the uniqueness in law for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ starting from X_0. Conversely the well-posedness of the martingale problem for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ implies the uniqueness in law for any X_0 of all the stochastic integro-differential equations (1.9) such that Σ satisfies (1.12).

1.2 Summary of contributions

This PhD thesis represents my work during the period 2008-2011, at the *Laboratoire de Probabilités et Modèles Aléatoires* (Université Pierre et Marie

Curie- Paris VI) under the supervision of Rama Cont. It studies various mathematical aspects of problems related to the Markovian projection of stochastic processes, and explores some applications of the results obtained to mathematical finance, in the context of semimartingale models.

Given a stochastic process ξ, modeled as a semimartingale, our aim is to build a Markov process X whose marginal laws are the same as ξ. This construction allows us to use analytical tools such as integro-differential equations to explore or compute quantities involving the marginal laws of ξ, even when ξ is not Markovian.

We present a systematic study of this problem from probabilistic viewpoint and from the analytical viewpoint. On the probabilistic side, given a discontinuous semimartingale we give an explicit construction of a Markov process X which mimics the marginal distributions of ξ, as the solution of a martingale problems for a certain integro-differential operator (Chapter 2). This construction extends the approach of Gyöngy to the discontinous case and applies to a wide range of examples which arise in applications, in particular in mathematical finance. Some applications are given in Chapters 4 and 5.

On the analytical side, we show that the flow of marginal distributions of a discontinuous semimartingale is the solution of an integro-differential equation, which extends the Kolmogorov forward equation to a non-Markovian setting. As an application, we derive a forward equation for option prices in a pricing model described by a discontinuous semimartingale (Chapter 3). This forward equation generalizes the Dupire equation, originally derived in the case of diffusion models, to the case of a discontinuous semimartingale.

1.2.1 Chapter 2 : Markovian projection of semimartingales

Consider, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, an Ito semimartingale, on the time interval $[0,T]$, $T>0$, given by the decomposition

$$\xi_t = \xi_0 + \int_0^t \beta_s\,ds + \int_0^t \delta_s\,dW_s + \int_0^t \int_{\|y\|\leq 1} y\,\tilde{M}(ds\,dy) + \int_0^t \int_{\|y\|>1} y\,M(ds\,dy),$$

where ξ_0 is in $\mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, M is an integer-valued random measure on $[0,T]\times\mathbb{R}^d$ with compensator measure μ

and $\tilde{M} = M - \mu$ is the compensated measure, β (resp. δ) is an adapted process with values in $\mathbb{R}^d$ (resp. $M_{d\times n}(\mathbb{R})$).

Chapter 2 is devoted to the construction of a Markov process X mimicking the marginals laws of the Itô semimartingale ξ_t. We propose a systematic construction of such a Markovian projection.

Our construction describes the mimicking Markov process X in terms of the local characteristics of the semimartingale ξ and we exhibit conditions under which the flow of marginal distributions of ξ can be matched by the Markov process X.

Namely, let us consider, for $(t,z) \in [0,T] \times \mathbb{R}^d, B \in \mathcal{B}(\mathbb{R}^d - \{0\})$,

$$\begin{aligned} b(t,z) &= \mathbb{E}\left[\beta_t | \xi_{t^-} = z\right], \\ a(t,z) &= \mathbb{E}\left[{}^t\delta_t \delta_t | \xi_{t^-} = z\right], \\ n(t,B,z) &= \mathbb{E}\left[m(.,t,B) | \xi_{t^-} = z\right]. \end{aligned} \tag{1.13}$$

If β, δ and m satisfy some boundedness and non-degeneracy conditions, namely either ${}^t\delta_t\delta_t$ is uniformly elliptic or ξ is a pure jump process such that its compensator has a singularity of the type α-stable in 0, and if b, a and n satisfy some continuity conditions, then one may identify the Markov process X (Theorem 2.2) as the unique solution of a martingale problem for the time-dependent integro-differential operator $\mathcal{L} = (\mathcal{L}_t)_{t\in[0,T]}$ defined, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, by

$$\begin{aligned} \mathcal{L}_t f(x) = b(t,x).\nabla f(x) &+ \sum_{i,j=1}^{d} \frac{a_{ij}(t,x)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &+ \int_{\mathbb{R}^d} [f(x+y) - f(x) - 1_{\{\|y\|\leq 1\}}\, y.\nabla f(x)] n(t,dy,x). \end{aligned} \tag{1.14}$$

Theses conditions are chosen with respect to the conditions given in Mikulevicus and Pragarauskas [77] for which the well-posedness holds for the martingale problem for the operator $\left((\mathcal{L}_t)_{t\in[0,T]}, \mathcal{C}_0^\infty(\mathbb{R}^d)\right)$ (see Proposition 2.1).

Our construction thus applies more readily to solutions of stochastic differential equations where the local characteristics are known but not the marginal distributions. One crucial key is the uniqueness result for the forward Kolmogorov equation associated to an integro-differential operator (Theorem 2.1). We use these results in section 2.2.4 to show in particular that the marginal distribution of ξ is the unique solution of an integro-differential equation : this is the Kolmogorov equation satisfied by the Markov process X mimicking ξ, extending thus the Kolmogorov forward equation to a non-Markovian setting and to discontinuous semimartingales (Theorem 2.3).

Section 2.3 shows how this result may be applied to processes whose jumps are represented as the integral of a predictable jump amplitude with respect to a Poisson random measure

$$\forall t \in [0,T] \qquad \zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int \psi_s(y)\, \tilde{N}(ds\, dy), \quad (1.15)$$

a representation often used in stochastic differential equations with jumps. In Section 2.4, we show that our construction may be applied to a large class of semimartingales, including smooth functions of a Markov process (Section 2.4.1), such as

$$f : \mathbb{R}^d \to \mathbb{R} \qquad \xi_t = f(Z_t)$$

for f regular enough and Z_t taken as the unique solution of a certain d-dimensional stochastic integro-differential equation, and time-changed Lévy processes (Section 2.4.2), such as

$$\xi_t = L_{\Theta_t} \qquad \Theta_t = \int_0^t \theta_s ds, \quad \theta_t > 0,$$

with L a scalar Lévy process with triplet (b, σ^2, ν).

1.2.2 Chapter 3: forward PIDEs for option pricing

The standard option pricing model of Black-Scholes and Merton [20, 75], widely used in option pricing, is known to be inconsistent with empirical observations on option prices and has led to many extensions, which include state-dependent volatility, multiple factors, stochastic volatility and jumps [30]. While more realistic from statistical point of view, these models increase the difficulty of calibration or pricing of options.

Since the seminal work of Black, Scholes and Merton [20, 75] partial differential equations (PDE) have been used as a way of characterizing and efficiently computing option prices. In the Black-Scholes-Merton model and various extensions of this model which retain the Markov property of the risk factors, option prices can be characterized in terms of solutions to a backward PDE, whose variables are time (to maturity) and the value of the underlying asset. The use of backward PDEs for option pricing has been extended to cover options with path-dependent and early exercise features, as well as to multifactor models (see e.g. [1]). When the underlying asset exhibit jumps, option prices can be computed by solving an analogous partial integro-differential equation (PIDE) [4, 31].

A second important step was taken by Dupire [33, 34, 36]. Let us recall that value $C_{t_0}(T, K)$ at t_0 of a call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T, K) = e^{-\int_{t_0}^{T} r(t)\, dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}]. \tag{1.16}$$

where $\mathbb{P}$ shall denote the risk-neutral measure. Dupire showed that when the underlying asset is assumed to follow a diffusion process

$$dS_t = S_t \sigma(t, S_t) dW_t,$$

prices of call options (at a given date t_0) solve a *forward* PDE

$$\frac{\partial C_{t_0}}{\partial T}(T, K) = -r(T)K \frac{\partial C_{t_0}}{\partial K}(T, K) + \frac{K^2 \sigma(T, K)^2}{2} \frac{\partial^2 C_{t_0}}{\partial K^2}(T, K)$$

on $[t_0, \infty[\times]0, \infty[$ in the strike and maturity variables, with the initial condition

$$\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

This forward equation allows to price call options with various strikes and maturities on the same underlying asset, by solving a *single* partial differential equation. Dupire's forward equation also provides useful insights into the *inverse problem* of calibrating diffusion models to observed call and put option prices [16].

Given the theoretical and computational usefulness of the forward equation, there have been various attempts to extend Dupire's forward equation to other types of options and processes, most notably to Markov processes with jumps [4, 26, 29, 62, 25]. Most of these constructions use the Markov property of the underlying process in a crucial way (see however [65]).

As noted by Dupire [35], one of the great advantages of the forward PDE is that it holds in a more general context than the backward PDE: even if the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale

$$dS_t = S_t \delta_t dW_t,$$

then call options still verify a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function $\sigma(t, S)$ given by

$$\sigma(t, S) = \sqrt{E[\delta_t^2 | S_t = S]}.$$

This method is linked to the "Markovian projection" problem: the construction of a Markov process which mimicks the marginal distributions of a martingale [15, 51, 74]. Such "mimicking processes" provide a method to extend the Dupire equation to non-Markovian settings.

We show in Chapter 3 that the forward equation for call prices holds in a more general setting, where the dynamics of the underlying asset is described by a – possibly discontinuous – semimartingale. Namely, we consider a (strictly positive) semimartingale S whose dynamics under the pricing measure $\mathbb{P}$ is given by

$$S_T = S_0 + \int_0^T r(t) S_{t^-} dt + \int_0^T S_{t^-} \delta_t dW_t + \int_0^T \int_{-\infty}^{+\infty} S_{t^-} (e^y - 1) \tilde{M}(dt\ dy), \tag{1.17}$$

where $r(t) > 0$ represents a (deterministic) bounded discount rate, δ_t the (random) volatility process and M is an integer-valued random measure with compensator $\mu(dt\, dy; \omega) = m(t, dy, \omega)\, dt$, representing jumps in the log-price, and $\tilde{M} = M - \mu$ is the compensated random measure associated to M. Both the volatility δ_t and $m(t, dy)$, which represents the intensity of jumps of size y at time t, are allowed to be stochastic. In particular, we do *not* assume the jumps to be driven by a Lévy process or a process with independent increments. The specification (1.17) thus includes most stochastic volatility models with jumps. Also, our derivation does not require ellipticity or non-degeneracy of the diffusion coefficient and under some integrability condition, we show that the call option price $(T, K) \mapsto C_{t_0}(T, K)$, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation

$$\begin{aligned} \frac{\partial C_{t_0}}{\partial T}(T, K) = -r(T) K \frac{\partial C_{t_0}}{\partial K}(T, K) + \frac{K^2 \sigma(T, K)^2}{2} \frac{\partial^2 C_{t_0}}{\partial K^2}(T, K) \\ + \int_0^{+\infty} y \frac{\partial^2 C_{t_0}}{\partial K^2}(T, dy)\, \chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right), \end{aligned} \tag{1.18}$$

on $[t_0, \infty[\times]0, \infty[$ with the initial condition: $\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} -$

$K)_+$, where ψ_t is the exponential double tail of the compensator $m(t,dy)$

$$\psi_t(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} m(t,du), & z < 0\ , \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} m(t,du), & z > 0, \end{cases} \tag{1.19}$$

and, for $t > t_0, z > 0$,

$$\begin{cases} \sigma(t,z) & = \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t-} = z\right]}, \\ \chi_{t,y}(z) & = \mathbb{E}\left[\psi_t\left(z\right)|S_{t-} = y\right]. \end{cases} \tag{1.20}$$

The results of Chapter 3 extend the forward equation from the original diffusion setting of Dupire [34] to various examples of non-Markovian and/or discontinuous processes. Our result implies previous derivations of forward equations [4, 26, 25, 29, 34, 35, 62, 73] as special cases. Section 3.2 gives examples of forward PIDEs obtained in various settings: time-changed Lévy processes, local Lévy models and point processes used in portfolio default risk modeling. These results are applicable to various stochastic volatility models with jumps, pure jump models and point process models used in equity and credit risk modeling.

Uniqueness of the solution of such PIDEs has been shown using analytical methods [8, 47] under various types of conditions on the coefficients, but which are difficult to verify in examples. We give a direct proof of the uniqueness of solutions for (1.18) using a probabilistic method, under explicit conditions which cover most examples of models used in finance. These conditions are closely related to the ones given in Chapter 2, ensuring the well-posedness for a martingale problem associated to a certain integro-differential operator.

In the case where the underlying risk factor follows an Itô process or a Markovian jump-diffusion driven by a Lévy process, we retrieve previously known forms of the forward equation. In this case, our approach gives a rigorous derivation of these results under precise assumptions, in a unified framework. In some cases, such as index options (Sec. 3.2.5) or CDO expected tranche notionals (Sec. 3.2.6), our method leads to a new, more general form of the forward equation valid for a larger class of models than previously studied [5, 29, 85].

The forward equation for call options is a PIDE in one (spatial) dimension, regardless of the number of factor driving the underlying asset. It may thus be used as a method for reducing the dimension of the problem. The case of index options (Section 3.2.5) in a multivariate jump-diffusion model illustrates how the forward equation projects a high dimensional pricing problem into a one-dimensional state equation, generalizing the forward PIDE studied by Avellaneda et al. [5] for the diffusion case.

1.2.3 Chapter 4 : short-time asymptotics for semimartingales

The result of chapters 2 and 3 reduce the computation of expectations of the type

$$\mathbb{E}\left[f(\xi_t)|\mathcal{F}_{t_0}\right] \tag{1.21}$$

to the computation of similar quantities when ξ is replaced by an appropriately chosen Markov process X, the Markovian projection of ξ. Chapter 4 uses similar ideas to compute analytically the asymptotics of such expressions as $t \to t_0$. Whereas for Markov process various well-known tools –partial differential equations, Monte Carlo simulation, semigroup methods– are available for the computation and approximation of conditional expectations, such tools do not carry over to the more general setting of semimartingales. Even in the Markov case, if the state space is high dimensional exact computations may be computationally prohibitive and there has been a lot of interest in obtaining approximations of (1.21) as $t \to t_0$. Knowledge of such *short-time asymptotics* is very useful not only for computation of conditional expectations but also for the estimation and calibration of such models. Accordingly, short-time asymptotics for (1.21) (which, in the Markov case, amounts to studying transition densities of the process ξ) has been previously studied for diffusion models [17, 18, 41], Lévy processes [60, 70, 83, 9, 44, 43, 91], Markov jump-diffusion models [3, 13] and one-dimensional martingales [78], using a variety of techniques. The proofs of these results in the case of Lévy processes makes heavy use of the independence of increments; proofs in other case rely on the Markov property, estimates for heat kernels for second-order differential operators or Malliavin calculus. What is striking, however, is the similarity of the results obtained in these different settings.

We reconsider here the short-time asymptotics of conditional expectations in a more general framework which contains existing models but allows to go beyond the Markovian setting and to incorporate path-dependent features. Such a framework is provided by the class of *Itô semimartingales*, which contains all the examples cited above but allows the use the tools of stochastic analysis. An *Itô semimartingale* on a filtered probability space

$(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic process ξ with the representation

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int_{\mathbb{R}^d} \kappa(y)\, \tilde{M}(dsdy) + \int_0^t \int_{\mathbb{R}^d} (y - \kappa(y))\, M(dsdy), \tag{1.22}$$

where ξ_0 is in $\mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, M is an integer-valued random measure on $[0, \infty] \times \mathbb{R}^d$ with compensator $\mu(\omega, dt, dy) = m(\omega, t, dy)dt$ and $\tilde{M} = M - \mu$ its compensated random measure, β (resp. δ) is an adapted process with values in $\mathbb{R}^d$ (resp. $M_{d\times n}(\mathbb{R})$) and

$$\kappa(y) = \frac{y}{1 + \|y\|^2}$$

is a truncation function.

We study the short-time asymptotics of conditional expectations of the form (1.21) where ξ is an Ito semimartingale of the form (1.22), for various classes of functions $f : \mathbb{R}^d \to \mathbb{R}$. First, we prove a general result for the case of $f \in C_b^2(\mathbb{R}^d, \mathbb{R})$. Then we will treat, when $d = 1$, the case of

$$\mathbb{E}\left[(\xi_t - K)^+ | \mathcal{F}_{t_0}\right], \tag{1.23}$$

which corresponds to the value at t_0 of a call option with strike K and maturity t in a model described by equation (4.2). We show that whereas the behavior of (4.3) in the case $K > \xi_{t_0}$ (*out-of-the-money* options) is linear in $t - t_0$, the asymptotics in the case $K = \xi_{t_0}$ (which corresponds to *at-the-money* options) depends on the fine structure of the semimartingale ξ at t_0. In particular, we show that for continuous semimartingales the short-maturity asymptotics of at-the-money options is determined by the local time of ξ at t_0. In each case we identify the leading term in the asymptotics and express this term in terms of the local characteristics of the semimartingale at t_0.

Our results unify various asymptotic results previously derived for particular examples of stochastic models and extend them to the more general case of a discontinuous semimartingale. In particular, we show that the independence of increments or the Markov property do not play any role in the derivation of such results.

Short-time asymptotics for expectations of the form (1.21) have been studied in the context of statistics of processes [60] and option pricing [3, 17, 18, 13, 44, 91, 78]. Berestycki, Busca and Florent [17, 18] derive short maturity asymptotics for call options when ξ_t is a diffusion, using analytical methods. Durrleman [38] studied the asymptotics of implied volatility in a general, non-Markovian stochastic volatility model. Jacod [60] derived asymptotics for (4.1) for various classes of functions f, when ξ_t is a Lévy process. Lopez [44] and Tankov [91] study the asymptotics of (4.3) when ξ_t is the exponential of a Lévy process. Lopez [44] also studies short-time asymptotic expansions for (4.1), by iterating the infinitesimal generator of the Lévy process ξ_t. Alos et al [3] derive short-maturity expansions for call options and implied volatility in a Heston model using Malliavin calculus. Benhamou et al. [13] derive short-maturity expansions for call options in a model where ξ is the solution of a Markovian SDE whose jumps are described by a compound Poisson process. These results apply to processes with independence of increments or solutions of a "Markovian" stochastic differential equation.

Durrleman studied the convergence of implied volatility to spot volatility in a stochastic volatility model with finite-variation jumps [37]. More recently, Nutz and Muhle-Karbe [78] study short-maturity asymptotics for call options in the case where ξ_t is a one-dimensional Itô semimartingale driven by a (one-dimensional) Poisson random measure whose Lévy measure is absolutely continuous. Their approach consists in "freezing" the characteristic triplet of ξ at t_0, approximating ξ_t by the corresponding Lévy process and using the results cited above [60, 44] to derive asymptotics for call option prices.

Our first contribution is to extend and unify these results to the more general case when ξ is a d-dimensional discontinuous semimartingale with jumps, described as in (1.22) in the case when $f \in C_b^2(\mathbb{R}^d, \mathbb{R})$: Theorem 4.1 gives a general result for the short-time asymptotics of $E[f(\xi_t)]$ in this setting. In contrast to previous derivations, our approach is purely based on Itô calculus, and makes no use of the Markov property or independence of increments. Also, our multidimensional setting allows to treat examples which are not accessible using previous results. For instance, when studying index options in jump-diffusion model (treated in the next chapter), one considers an index $I_t = \sum w_i S_t^i$ where $(S^1, ..., S^d)$ are Itô semimartingales. In this framework, I is indeed an Itô semimartingale whose stochastic integral representation is implied by those of S^i but it is naturally represented in terms of a d-dimensional integer-valued random measure, not a one-dimensional Poisson random measure. Our setting provides a natural framework for treating such examples.

As an application, we derive the asymptotic behavior of call option prices close to maturity, in the pricing model described by equation (1.17). We show that while the value of *out of the money* options is linear in time-to-maturity, *at the money* options have a different asymptotic behavior. Indeed, as already noted in the case of Lévy processes by Lopez [44] and Tankov [91], the short maturity behavior of at-the-money options depends on the presence of a continuous martingale component and, in absence of such a component, on the degree of activity of small jumps, measured by the singularity of the Lévy measure at zero. We will show here that similar results hold in the semimartingale case. We distinguish three cases:

1. S is a pure jump process of finite variation: in this case at the money call options behave linearly in t when $t - t_0 \to 0$ (Theorem 4.4).

2. S is a pure jump process of infinite variation and its small jumps resemble those of an α-stable process: in this case at the money call options have an asymptotic behavior of order $|t - t_0|^{1/\alpha}$ (Theorem 4.5).

3. S has a continuous martingale component which is non-degenerate in the neighborhood of t_0: in this case at the money call options are of order $\sqrt{t - t_0}$ as $t \to t_0$, whether or not jumps are present (Theorem 4.3).

We observe that, contrarily to the case of out-of-the money options where the presence of jumps dominates the asymptotic behavior, for at-the-money options the presence or absence of a continuous martingale (Brownian) component dominates the asymptotic behavior. Our approach highlights the connection between the asymptotics of at-the-money options and the behavior of the local time of the semimartingale S_t at S_0.

These results generalize and extend various asymptotic results previously derived for diffusion models [17], Lévy processes [60, 44, 91], Markov jump-diffusion models [13] and one-dimensional martingales [37, 38, 78] to the more general case of a discontinous, $\mathbb{R}^d-$valued semimartingale.

1.2.4 Chapter 5 : application to index options

Consider a multi-asset market with d assets, whose prices $S^1, ..., S^d$ are represented as Ito semimartingales:

$$S_t^i = S_0^i + \int_0^t r(s) S_{s^-}^i ds + \int_0^t S_{s^-}^i \delta_s^i dW_s^i + \int_0^t \int_{\mathbb{R}^d} S_{s^-}^i (e^{y_i} - 1) \tilde{N}(ds\, dy),$$

where

- δ^i is an adapted process taking values in $\mathbb{R}$ representing the volatility of the asset i, W is a d-dimensional Wiener process : for all $1 \leq (i,j) \leq d$, $\langle W^i, W^j \rangle_t = \rho_{ij} t$,
- N is a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with compensator $\nu(dy)\,dt$,
- $\tilde{N}$ denotes its compensated random measure.

We consider an index, defined as a weighted sum of the asset prices:

$$I_t = \sum_{i=1}^{d} w_i S_t^i, \quad d \geq 2.$$

The pricing of index options involves the computation of quantities of the form $\mathbb{E}\left[f(I_t)|\mathcal{F}_{t_0}\right]$ and Chapter 4 shows that the short time asymptotics for these quantitie that we have characterized explicitly in terms of the characteristic triplet of the discontinuous semimartingale I_t in Chapter 3.

Short time asymptotics of index call option prices have been computed by Avellaneda & al [5] in the case where S is a continuous process. Results of Chapter 4 show that this asymptotic behavior is quite diferent for at the money or out of the money options. At the money options exhibit a bahavior in $O(\sqrt{t})$ which involves the diffusion component of I_t whereas out of the money options exhibit a linear behavior in t which only involves the jumps of I_t.

In this Chapter, we propose an analytical approximation for short maturity index options, generalizing the approach by Avellaneda & al. [5] to the multivariate jump-diffusion case. We implement this method in the case of the Merton model in dimension $d = 2$ and $d = 30$ and study its numerical precision.

The main difficulty is that, even when the joint dynamics of the index components $(S^1, ..., S^d)$ is Markovian, the index I_t is not a Markov process but only a semimartingale. The idea is to consider the *Markovian projection* of the index process, an auxiliary Markov process which has the same marginals as I_t, and use it to derive the asymptotics of index options, using the results of Chapter 4. This approximation is shown to depend only on the coefficients of this Markovian projection, so the problem boils down to computing effectively these coefficients: the local volatility function and the 'effective Lévy measure', defined as the conditional expectation given I_t of the jump compensator of I.

The computation of the effective Lévy measure involves a d-dimensional integral. Computing directly this integral would lead, numerically speaking, to a complexity increasing exponentially with d. We propose different techniques to simplify this computation and make it feasible when the dimension is large, using the Laplace method to approximate the exponential double tail of the jump measure of I_t. Laplace method is an important tool when one wants to approximate consistently high-dimensional integrals and avoids a numerical exponential complexity increasing with the dimension. Avellaneda & al [5] use this method in the diffusion case to compute the local volatility of an index option by using a steepest descent approximation, that is by considering that, for t small enough, the joint law of $(S^1, \cdots, S^d)$ given

$$\{\sum_{i=1}^{d} w_i S_t^i = u\},$$

is concentrated around the most probable path, which we proceed to identify.

1.2.5 List of publications and working papers

1. A. Bentata, R. Cont (2009) *Forward equations for option prices in semimartingale models*, to appear in **Finance & Stochastics**. http://arxiv.org/abs/1001.1380
2. A. Bentata, R. Cont (2009) *Mimicking the marginal distributions of a semimartingale*, submitted. http://arxiv.org/abs/0910.3992
3. A. Bentata, R. Cont (2011) *Short-time asymptotics of marginal distributions of a semimartingale*, submitted.
4. A. Bentata (2008) A note about conditional Ornstein-Uhlenbeck processes, http://arxiv.org/abs/0801.3261.
5. A. Bentata, M. Yor (2008) From Black-Scholes and Dupire formulae to last passage times of local martingales. Part A : The infinite time horizon. http://arxiv.org/abs/0806.0239.
6. A. Bentata, M. Yor (2008) From Black-Scholes and Dupire formulae to last passage times of local martingales. Part B : The finite time horizon. http://arxiv.org/abs/0807.0788.

Chapter 2

Markovian projection of semimartingales

We exhibit conditions under which the flow of marginal distributions of a discontinuous semimartingale ξ can be matched by a Markov process, whose infinitesimal generator is expressed in terms of the local characteristics of ξ. Our construction applies to a large class of semimartingales, including smooth functions of a Markov process. We use this result to derive a partial integro-differential equation for the one-dimensional distributions of a semimartingale, extending the Kolmogorov forward equation to a non-Markovian setting.

2.1 Introduction

Stochastic processes with path-dependent / non-Markovian dynamics used in various fields such as physics and mathematical finance present challenges for computation, simulation and estimation. In some applications where one is interested in the marginal distributions of such processes, such as option pricing or Monte Carlo simulation of densities, the complexity of the model can be greatly reduced by considering a low-dimensional Markovian model with the same marginal distributions. Given a process ξ, a Markov process X is said to *mimick* ξ on the time interval $[0,T]$, $T>0$, if ξ and X have the same marginal distributions:

$$\forall t \in [0,T], \qquad \xi_t \overset{d}{=} X_t. \tag{2.1}$$

The construction of such mimicking process was first suggested by Brémaud [21] in the context of queues. Construction of mimicking processes of 'Markovian' type has been explored for Ito processes [51] and marked point processes [28]. A notable application is the derivation of forward equations for option pricing [14, 34].

We propose in this paper a systematic construction of such such mimicking processes for (possibly discontinuous) semimartingales. Given a semimartingale ξ, we give conditions under which there exists a Markov process X whose marginal distributions are identical to those of ξ, and give an explicit construction of the Markov process X as the solution of a martingale problem for an integro-differential operator [10, 66, 88, 89].

Moreover, we show that our construction may be seen as a *projection* on the set of Markov processes.

In the martingale case, the Markovian projection problem is related to the problem of constructing martingales with a given flow of marginals, which dates back to Kellerer [64] and has been recently explored by Yor and coauthors [7, 55, 74] using a variety of techniques. The construction proposed in this paper is different from the others since it does not rely on the martingale property of ξ. We shall see nevertheless that our construction preserves the (local) martingale property. Also, whereas the approaches described in [7, 55, 74] use as a starting point the marginal distributions of ξ, our construction describes the mimicking Markov process X in terms of the local characteristics [61] of the semimartingale ξ. Our construction thus applies more readily to solutions of stochastic differential equations where the local characteristics are known but not the marginal distributions.

Section 2.2 presents a Markovian projection result for a $\mathbb{R}^d$-valued semimartingale given by its local characteristics. We use these results in section 2.2.4 to derive a partial integro-differential equation for the one-dimensional distributions of a discontinuous semimartingale, thus extending the Kolmogorov forward equation to a non-Markovian setting. Section 2.3 shows how this result may be applied to processes whose jumps are represented as the integral of a predictable jump amplitude with respect to a Poisson random measure, a representation often used in stochastic differential equations with jumps. In Section 2.4 we show that our construction applies to a large class of semimartingales, including smooth functions of a Markov process (Section 2.4.1), and time-changed Lévy processes (Section 2.4.2).

2.2 A mimicking theorem for discontinuous semimartingales

Consider, on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$, an Ito semimartingale, given by the decomposition

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int_{\|y\|\leq 1} y\, \tilde{M}(ds\, dy) + \int_0^t \int_{\|y\|>1} y\, M(ds\, dy), \tag{2.2}$$

where ξ_0 is in $\mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, M is an integer-valued random measure on $[0,\infty] \times \mathbb{R}^d$ with compensator measure μ and $\tilde{M} = M - \mu$ is the compensated measure [61, Ch.II,Sec.1], β (resp. δ) is an adapted process with values in $\mathbb{R}^d$ (resp. $M_{d\times n}(\mathbb{R})$).

Let $\Omega_0 = D([0,T], \mathbb{R}^d)$ be the Skorokhod space of right-continuous functions with left limits. Denote by $X_t(\omega) = \omega(t)$ the canonical process on Ω_0, $\mathcal{B}_t^0$ its natural filtration and $\mathcal{B}_t \equiv \mathcal{B}_{t+}^0$.

Our goal is to construct a probability measure $\mathbb{Q}$ on Ω_0 such that X is a Markov process under $\mathbb{Q}$ and ξ and X have the same one-dimensional distributions:

$$\forall t \in [0,T], \quad \xi_t \overset{\mathrm{d}}{=} X_t.$$

In order to do this, we shall characterize $\mathbb{Q}$ as the solution of a *martingale problem* for an appropriately chosen integro-differential operator $\mathcal{L}$.

2.2.1 Martingale problems for integro-differential operators

Let $\mathcal{C}_b^0(\mathbb{R}^d)$ denote the set of bounded and continuous functions on $\mathbb{R}^d$, $\mathcal{C}_0^\infty(\mathbb{R}^d)$ the set of infinitely differentiable functions with compact support on $\mathbb{R}^d$ and $\mathcal{C}_0(\mathbb{R}^d)$ the set of continuous functions defined on $\mathbb{R}^d$ and vanishing at infinity. Let $\mathcal{R}(\mathbb{R}^d - \{0\})$ denote the space of Lévy measures on $\mathbb{R}^d$ i.e. the set of non-negative σ-finite measures ν on $\mathbb{R}^d - \{0\}$ such that

$$\int_{\mathbb{R}^d - \{0\}} \nu(dy) \left(1 \wedge \|y\|^2\right) < \infty.$$

Consider a time-dependent integro-differential operator $\mathcal{L} = (\mathcal{L}_t)_{t\in[0,T]}$

defined, for $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, by

$$\begin{aligned}\mathcal{L}_t f(x) &= b(t,x).\nabla f(x) + \sum_{i,j=1}^{d} \frac{a_{ij}(t,x)}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d} [f(x+y) - f(x) - 1_{\{\|y\|\leq 1\}}\, y.\nabla f(x)] n(t,dy,x),\end{aligned} \tag{2.3}$$

where $a : [0,T] \times \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$, $b : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ and $n : [0,T] \times \mathbb{R}^d \mapsto \mathcal{R}(\mathbb{R}^d - \{0\})$ are measurable functions.

For $(t_0, x_0) \in [0,T] \times \mathbb{R}^d$, we recall that a probability measure $\mathbb{Q}_{t_0,x_0}$ on $(\Omega_0, \mathcal{B}_T)$ is a solution to the *martingale problem* for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ if $\mathbb{Q}\left(X_u = x_0,\, 0 \leq u \leq t_0\right) = 1$ and for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, the process

$$f(X_t) - f(x_0) - \int_{t_0}^{t} \mathcal{L}_s f(X_s)\, ds$$

is a $(\mathbb{Q}_{t_0,x_0}, (\mathcal{B}_t))$-martingale on $[0,T]$. Existence, uniqueness and regularity of solutions to martingale problems for integro-differential operators have been studied under various conditions on the coefficients [90, 59, 40, 66, 77, 42].

We make the following assumptions on the coefficients:

Assumption 2.1 (Boundedness of coefficients)**.**

$$(i) \quad \exists K_1 > 0, \quad \forall (t,z) \in [0,T] \times \mathbb{R}^d, \quad \|b(t,z)\| + \|a(t,z)\| + \int \left(1 \wedge \|y\|^2\right) n(t,dy,z) \leq K_1$$

$$(ii) \qquad \lim_{R\to\infty} \int_0^T \sup_{z\in\mathbb{R}^d} n\left(t, \{\|y\| \geq R\}, z\right)\, dt = 0.$$

where $\|.\|$ denotes the Euclidean norm.

Assumption 2.2 (Continuity)**.**
(i) For $t \in [0,T]$ and $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$, $b(t,.)$, $a(t,.)$ and $n(t,B,.)$ are continuous on $\mathbb{R}^d$, uniformly in $t \in [0,T]$.
(ii) For all $z \in \mathbb{R}^d$, $b(.,z)$, $a(.,z)$ and $n(.,B,z)$ are continuous on $[0,T[$, uniformly in $z \in \mathbb{R}^d$.

Assumption 2.3 (Non-degeneracy).

$$\text{Either} \qquad \forall R > 0\, \forall t \in [0,T] \quad \inf_{\|z\| \le R}\ \inf_{x \in \mathbb{R}^d,\, \|x\|=1} {}^t x.a(t,z).x > 0$$

or $\quad a \equiv 0 \quad$ and there exists $\quad \beta \in]0,2[, C > 0, K_2 > 0$, and a family $n^\beta(t,dy,z)$ of positive measures such that

$$\forall (t,z) \in [0,T] \times \mathbb{R}^d \qquad n(t,dy,z) = n^\beta(t,dy,z) + 1_{\{\|y\| \le 1\}} \frac{C}{\|y\|^{d+\beta}}\, dy,$$

$$\int \left(1 \wedge \|y\|^\beta\right) n^\beta(t,dy,z) \le K_2, \quad \lim_{\epsilon \to 0} \sup_{z \in \mathbb{R}^d} \int_{\|y\| \le \epsilon} \|y\|^\beta\, n^\beta(t,dy,z) = 0.$$

Mikulevicius and Pragarauskas [77, Theorem 5] show that if $\mathcal{L}$ satisfies Assumptions 2.1, 2.2 and 2.3 (which corresponds to a "non-degenerate Lévy operator" in the terminology of [77]) the martingale problem for $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d)\,)$ has a unique solution $\mathbb{Q}_{t_0,x_0}$ for every initial condition $(t_0,x_0) \in [0,T] \times \mathbb{R}^d$:

Proposition 2.1. *Under Assumptions 2.1, 2.2 and 2.3 the martingale problem for $((\mathcal{L}_t)_{t\in[0,T]}, \mathcal{C}_0^\infty(\mathbb{R}^d))$ on $[0,T]$ is well-posed : for any $x_0 \in \mathbb{R}^d, t_0 \in [0,T]$, there exists a unique probability measure $\mathbb{Q}_{t_0,x_0}$ on $(\Omega_0, \mathcal{B}_T)$ such that $\mathbb{Q}\left(X_u = x_0,\, 0 \le u \le t_0\right) = 1$ and for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,*

$$f(X_t) - f(x_0) - \int_{t_0}^t \mathcal{L}_s f(X_s)\, ds$$

is a $(\mathbb{Q}_{t_0,x_0}, (\mathcal{B}_t)_{t\ge 0})$-martingale on $[0,T]$. Under $\mathbb{Q}_{t_0,x_0}$, (X_t) is a Markov process and the evolution operator $(Q_{t_0,t})_{t\in[t_0,T]}$ defined by

$$\forall f \in \mathcal{C}_b^0(\mathbb{R}^d), \quad Q_{t_0,t} f(x_0) = \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[f(X_t)\right] \tag{2.4}$$

verifies the following continuity property:

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad \lim_{t \downarrow t_0} Q_{t_0,t} f(x_0) = f(x_0). \tag{2.5}$$

In particular, denoting $q_{t_0,t}(x_0,dy)$ the marginal distribution of X_t, the map

$$t \in [t_0, T[\mapsto \int_{\mathbb{R}^d} q_{t_0,t}(x_0,dy) f(y) \tag{2.6}$$

is right-continuous, for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$.

Proof. By a result of Mikulevicius and Pragarauskas [77, Theorem 5], the martingale problem is well-posed. We only need to prove that the continuity property (2.5) holds on $[t_0, T[$ for any $x_0 \in \mathbb{R}^d$. For $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} Q_{t_0,t}f(x_0) &= \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[f(X_t)\right] \\ &= f(x_0) + \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[\int_{t_0}^t \mathcal{L}_s f(X_s)\, ds\right]. \end{aligned}$$

Given Assumption 2.1, $t \in [t_0, T] \mapsto \int_0^t \mathcal{L}_s f(X_s)\, ds$ is uniformly bounded on $[t_0, T]$. By Assumption 2.2, since X is right continuous, $s \in [t_0, T[\mapsto \mathcal{L}_s f(X_s)$ is right-continuous up to a $\mathbb{Q}_{t_0,x_0}$-null set and

$$\lim_{t \downarrow t_0} \int_{t_0}^t \mathcal{L}_s f(X_s)\, ds = 0 \quad \text{a.s.}$$

Applying the dominated convergence theorem yields,

$$\lim_{t \downarrow t_0} \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[\int_{t_0}^t \mathcal{L}_s f(X_s)\, ds\right] = 0,$$

that is

$$\lim_{t \downarrow t_0} Q_{t_0,t}f(x_0) = f(x_0),$$

implying that $t \in [t_0, T[\mapsto Q_{t_0,t}f(x_0)$ is right-continuous at t_0. □

2.2.2 A uniqueness result for the Kolmogorov forward equation

An important property of continuous-time Markov processes is their link with partial (integro-)differential equation (PIDE) which allows to use analytical tools for studying their probabilistic properties. In particular the transition density of a Markov process solves the forward Kolmogorov equation (or Fokker-Planck equation) [89]. The following result shows that under Assumptions 2.1, 2.2 and 2.3 the forward equation corresponding to $\mathcal{L}$ has a unique solution:

Theorem 2.1 (Kolmogorov Forward equation)**.** *Under Assumptions 2.1, 2.2 and 2.3, for each $(t_0, x_0) \in [0, T] \times \mathbb{R}^d$, there exists a unique family $(p_{t_0,t}(x_0, dy), t \geq t_0)$ of positive bounded measures on $\mathbb{R}^d$ such that $\forall t \geq$*

$t_0, \forall g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} p_{t_0,t}(x_0, dy) g(y) = g(x_0) + \int_{t_0}^t \int_{\mathbb{R}^d} p_{t_0,s}(x_0, dy) \mathcal{L}_s g(y)\, ds, \tag{2.7}$$
$$p_0(x_0, .) = \epsilon_{x_0},$$

where ϵ_{x_0} *is the point mass at* x_0.

$p_{t_0,t}(x_0, .)$ *is the distribution of* X_t, *where* $(X, \mathbb{Q}_{t_0,x_0})$ *is the unique solution of the martingale problem for* $(\mathcal{L}, \mathcal{C}_0^\infty(\mathbb{R}^d)\ )$.

Equation (2.7) is the weak form of the forward Kolmogorov equation for the time-dependent operator $(\mathcal{L}_t)_{t\geq 0}$.

Proof. 1. Under Assumptions 2.1, 2.2 and 2.3, Proposition 2.1 implies that the martingale problem for $\mathcal{L}$ on the domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is well-posed. Denote $(X, \mathbb{Q}_{t_0,x_0})$ the unique solution of the martingale problem for $\mathcal{L}$ with initial condition $x_0 \in \mathbb{R}^d$ at t_0, and define

$$\forall t \geq t_0, \quad \forall g \in \mathcal{C}_b^0(\mathbb{R}^d), \quad Q_{t_0,t} g(x_0) = \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[g(X_t)\right]. \tag{2.8}$$

By [77, Theorem 5], $(Q_{s,t}, 0 \leq s \leq t)$ is then a (time-inhomogeneous) semigroup, satisfying the continuity property (2.5).

If $q_{t_0,t}(x_0, dy)$ denotes the law of X_t under $\mathbb{Q}_{t_0,x_0}$, the martingale property implies that $q_{t_0,t}(x_0, dy)$ satisfies

$$\forall g \in \mathcal{C}_0^\infty(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} q_{t_0,t}(x_0, dy) g(y) = g(x_0) + \int_{t_0}^t \int_{\mathbb{R}^d} q_{t_0,s}(x_0, dy) \mathcal{L}_s g(y)\, ds. \tag{2.9}$$

Proposition 2.1 provides the right-continuity of $t \mapsto \int_{\mathbb{R}^d} q_{t_0,t}(x_0, dy) g(y)$ for any $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$. Given Assumption 2.2, $q_{t_0,t}$ is a solution of (2.7) with initial condition $p_{t_0,t_0}(x_0, .) = \epsilon_{x_0}$ and has unit mass.

To show uniqueness of solutions of (2.7), we will rewrite (2.7) as the forward Kolmogorov equation associated with a *homogeneous* operator on space-time domain and use uniqueness results for the corresponding homogeneous equation.

2. Let $\mathcal{D}^0 \equiv \mathcal{C}_0^1([0, \infty[\otimes \mathcal{C}_0^\infty(\mathbb{R}^d)$ be the (algebraic) tensor product of $\mathcal{C}_0^1([0, \infty[)$ and $\mathcal{C}_0^\infty(\mathbb{R}^d)$. Define the operator A on $\mathcal{D}^0$ by

$$\forall f \in C_0^\infty(\mathbb{R}^d), \forall \gamma \in \mathcal{C}_0^1([0, \infty[), \quad A(f\gamma)(t, x) = \gamma(t)\mathcal{L}_t f(x) + f(x)\gamma'(t). \tag{2.10}$$

[40, Theorem 7.1, Chapter 4] implies that for any $x_0 \in \mathbb{R}^d$, if $(X, \mathbb{Q}_{t_0,x_0})$ is a solution of the martingale problem for $\mathcal{L}$, then the law of $\eta_t = (t, X_t)$ under $\mathbb{Q}_{t_0,x_0}$ is a solution of the martingale problem for A: in particular for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, $\gamma \in \mathcal{C}^1([0,\infty)\ )$ and $\forall t \geq t_0$,

$$\int q_{t_0,t}(x_0, dy) f(y)\gamma(t) = f(x_0)\gamma(0) + \int_{t_0}^t \int q_{t_0,s}(x_0, dy) A(f\gamma)(s, y)\, ds. \tag{2.11}$$

[40, Theorem 7.1, Chapter 4] implies also that if the law of $\eta_t = (t, X_t)$ is a solution of the martingale problem for A then the law of X is also a solution of the martingale problem for $\mathcal{L}$, namely: uniqueness holds for the martingale problem associated to the operator $\mathcal{L}$ on $\mathcal{C}_0^\infty(\mathbb{R}^d)$ if and only if uniqueness holds for the martingale problem associated to the martingale problem for A on $\mathcal{D}^0$. Denote $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ the set of continuous functions on $[0,\infty) \times \mathbb{R}^d$ and vanishing at infinity. Define, for $t \geq 0$ and $h \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$,

$$\forall (s,x) \in [0,\infty[\times\mathbb{R}^d, \quad \mathcal{U}_t h(s,x) = Q_{s,s+t}\left(h(t+s,.)\right)(x). \tag{2.12}$$

The properties of $Q_{s,t}$ then imply that $(\mathcal{U}_t, t \geq 0)$ is a family of linear operators on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ satisfying $\mathcal{U}_t\mathcal{U}_r = \mathcal{U}_{t+r}$ on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ and $\mathcal{U}_t h \to h$ as $t \downarrow 0$ on $\mathcal{D}^0$. $(\mathcal{U}_t, t \geq 0)$ is thus a contraction semigroup on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ satisfying a continuity property on $\mathcal{D}^0$:

$$\forall h \in \mathcal{D}^0, \quad \lim_{t\downarrow\epsilon} \mathcal{U}_t h(s,s) = \mathcal{U}_\epsilon h(s,x). \tag{2.13}$$

3. We apply [40, Theorem 2.2, Chapter 4] to prove that $(\mathcal{U}_t, t \geq 0)$ is a strongly continuous contraction on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ with infinitesimal generator given by the closure $\overline{A}$ of A. First, observe that $\mathcal{D}^0$ is dense in $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$. The well-posedness of the martingale problem for A implies that A satisfies the maximum principle. It is thus sufficient to prove that $Im(\lambda - \overline{A})$ is dense in $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ for some λ where $Im(\lambda - \overline{A})$ denotes the image of $\mathcal{D}_0$ under $(\lambda - \overline{A})$.

4. Without loss of generality, let us put $t_0 = 0$ in the sequel. For $h \in \mathcal{D}^0$, the martingale property yields

$$\begin{gathered}\forall 0 \leq \epsilon \leq t < T,\ \forall (s,x) \in [0,T] \times \mathbb{R}^d, \\ \mathcal{U}_t h(s,x) - \mathcal{U}_\epsilon h(s,x) = \int_\epsilon^t \mathcal{U}_u A h(s,x)\, du.\end{gathered} \tag{2.14}$$

which yields in turn

$$\begin{aligned}\int_\epsilon^T e^{-t}\mathcal{U}_t h\,dt &= \int_\epsilon^T e^{-t}\mathcal{U}_\epsilon h\,dt + \int_\epsilon^T e^{-t}\int_\epsilon^t \mathcal{U}_u Ah\,du\,dt \\ &= \mathcal{U}_\epsilon h\left[e^{-\epsilon} - e^{-T}\right] + \int_\epsilon^T du\left(\int_u^T e^{-t}\,dt\right)\mathcal{U}_u Ah \\ &= \mathcal{U}_\epsilon h\left[e^{-\epsilon} - e^{-T}\right] + \int_\epsilon^T du\,\left[e^{-u} - e^{-T}\right]\mathcal{U}_u Ah \\ &= e^{-\epsilon}\mathcal{U}_\epsilon h - e^{-T}\left[\mathcal{U}_\epsilon h + \int_\epsilon^T \mathcal{U}_u Ah\,du\right] + \int_\epsilon^T du\,e^{-u}\mathcal{U}_u Ah.\end{aligned}$$

Using (2.14) and gathering all the terms together yields,

$$\int_\epsilon^T e^{-t}\mathcal{U}_t h\,dt = e^{-\epsilon}\mathcal{U}_\epsilon h - e^{-T}\mathcal{U}_T h + \int_\epsilon^T du\,e^{-u}\mathcal{U}_u Ah. \tag{2.15}$$

Let us focus on the quantity

$$\int_\epsilon^T du\,e^{-u}\mathcal{U}_u Ah.$$

Observing that

$$\frac{1}{\epsilon}\left[\mathcal{U}_{t+\epsilon}h - \mathcal{U}_t h\right] = \frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]\mathcal{U}_t h = \mathcal{U}_t\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h.$$

Since $h \in \mathrm{dom}(A)$, taking $\epsilon \to 0$ yields

$$\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h \to Ah$$

Given Assumptions 2.1 and 2.2, $Ah \in \mathcal{C}_b^0([0,\infty[\times\mathbb{R}^d)$ and the contraction property of $\mathcal{U}$ yields,

$$\|\mathcal{U}_t\left(\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h - Ah\right)\| \leq \|\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h - Ah\|,$$

where $\|.\|$ denotes the supremum norm on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$. Thus

$$\lim_{\epsilon\to 0}\mathcal{U}_t\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h = \mathcal{U}_t Ah$$

Hence, the limit when $\epsilon \to 0$ of

$$\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]\mathcal{U}_t h$$

exists, implying that $\mathcal{U}_t h$ belongs to the domain of $\overline{A}$ for any $h \in \mathcal{D}^0$. Thus,

$$\int_\epsilon^T du\, e^{-u}\,\mathcal{U}_u h$$

belongs to the domain of $\overline{A}$ and

$$\int_\epsilon^T du\, e^{-u}\,\mathcal{U}_u A h = \overline{A}\int_\epsilon^T du\, e^{-u}\,\mathcal{U}_u h.$$

Since $\mathcal{U}$ is a contraction semigroup and given the continuity property of $\mathcal{U}_t$ on the space $\mathcal{D}^0$, one may take $\epsilon \to 0$ and $T \to \infty$ in (2.15), leading to

$$\int_0^\infty e^{-t}\mathcal{U}_t h\, dt = \mathcal{U}_0 + \overline{A}\int_0^\infty du\, e^{-u}\,\mathcal{U}_u h.$$

Thus

$$\left(I - \overline{A}\right)\int_0^\infty du\, e^{-u}\,\mathcal{U}_u h(s,x) = \mathcal{U}_0 h(s,x) = h(s,x),$$

yielding $h \in Im(I - \overline{A})$. We have shown that $(\mathcal{U}_t, t \geq 0)$ generates a strongly continuous contraction on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ with infinitesimal generator $\overline{A}$ (see [40, Theorem 2.2, Chapter 4]). The Hille-Yosida theorem [40, Proposition 2.6, Chapter 1] then implies that for all $\lambda > 0$

$$Im(\lambda - \overline{A}) = \mathcal{C}_0([0,\infty[\times\mathbb{R}^d).$$

5. Now let $p_t(x_0, dy)$ be another solution of (2.7). First, considering equation (3.13) for the particular function $g(y) = 1$, yields

$$\forall t \geq 0 \quad \int_{\mathbb{R}^d} p_t(x_0, dy) = 1,$$

and $p_t(x_0, dy)$ has mass 1.

Then, an integration by parts implies that, for $(f,\gamma) \in \mathcal{C}_0^\infty(\mathbb{R}^d) \times \mathcal{C}_0^1([0,\infty[)$,

$$\int_{\mathbb{R}^d} p_t(x_0,dy) f(y)\gamma(t) = f(x_0)\gamma(0) + \int_0^t \int_{\mathbb{R}^d} p_s(x_0,dy) A(f\gamma)(s,y)\, ds. \tag{2.16}$$

Define, for $h \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$,

$$\forall (t,x_0) \in [0,\infty[\times\mathbb{R}^d, \qquad \mathcal{P}_t h(0,x_0) \;=\; \int_{\mathbb{R}^d} p_t(x_0,dy) h(t,y).$$

Using (2.16) we have, for $(f,\gamma) \in \mathcal{C}_0^\infty(\mathbb{R}^d) \times \mathcal{C}_0^1([0,\infty[)$,

$$\forall \epsilon > 0 \quad \mathcal{P}_t(f\gamma) - \mathcal{P}_\epsilon(f\gamma) = \int_\epsilon^t \int_{\mathbb{R}^d} p_u(dy) A(f\gamma)(u,y)\, du = \int_\epsilon^t \mathcal{P}_u(A(f\gamma))\, du, \tag{2.17}$$

and by linearity, for any $h \in \mathcal{D}^0$,

$$\mathcal{P}_t h - \mathcal{P}_\epsilon h = \int_\epsilon^t \mathcal{P}_u A h\, du. \tag{2.18}$$

Multiplying by $e^{-\lambda t}$ and integrating with respect to t we obtain, for $\lambda > 0$,

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t}\, \mathcal{P}_t h(0,x_0)\, dt &= h(0,x_0) + \lambda \int_0^\infty e^{-\lambda t} \int_0^t \mathcal{P}_u(Ah)(0,x_0)\, du\, dt \\
&= h(0,x_0) + \lambda \int_0^\infty \left(\int_u^\infty e^{-\lambda t} dt \right) \mathcal{P}_u(Ah)(0,x_0)\, du \\
&= h(0,x_0) + \int_0^\infty e^{-\lambda u}\, \mathcal{P}_u(Ah)(0,x_0)\, du.
\end{aligned}$$

Similarly, we obtain for any $\lambda > 0$,

$$\lambda \int_0^\infty e^{-\lambda t}\, \mathcal{U}_t h(0,x_0)\, dt \;=\; h(0,x_0) + \int_0^\infty e^{-\lambda u}\, \mathcal{U}_u(Ah)(0,x_0)\, du.$$

Hence for any $h \in \mathcal{D}^0$, we have

$$\int_0^\infty e^{-\lambda t}\, \mathcal{U}_t(\lambda - A) h(0,x_0)\, dt = h(0,x_0) = \int_0^\infty e^{-\lambda t}\, \mathcal{P}_t(\lambda - A) h(0,x_0)\, dt. \tag{2.19}$$

Using the density of $Im(\lambda - A)$ in $Im(\lambda - \overline{A}) = \mathcal{C}_0([0, \infty \times \mathbb{R}^d)$, we get

$$\forall g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^d), \quad \int_0^\infty e^{-\lambda t}\,\mathcal{U}_t g\,(0,x_0)\,dt = \int_0^\infty e^{-\lambda t}\,\mathcal{P}_t g(0,x_0)\,dt, \tag{2.20}$$

so the Laplace transform of $t \mapsto \mathcal{P}_t g\,(0,x_0)$ is uniquely determined. Using (2.18), for any $h \in \mathcal{D}^0$, $t \mapsto \mathcal{P}_t h(0,x_0)$ is right-continuous:

$$\forall h \in \mathcal{D}^0, \qquad \lim_{t\downarrow\epsilon} \mathcal{P}_t h(0,x_0) = \mathcal{P}_\epsilon h(0,x_0).$$

Furthermore, the density of $\mathcal{D}_0$ in $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ implies the weak-continuity of $t \to \mathcal{P}_t g(0,x_0)$ for any $g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$. Indeed, let $g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$, there exists $(h_n)_{n\geq 0} \in \mathcal{D}_0$ such that

$$\lim_{n\to\infty} \|g - h_n\| = 0$$

Then equation (2.18) yields,

$$\begin{aligned}
& |\mathcal{P}_t g(0,x_0) - \mathcal{P}_\epsilon g(0,x_0)| \\
= \;& |\mathcal{P}_t(g-h_n)(0,x_0) + (\mathcal{P}_t - \mathcal{P}_\epsilon)\,h_n(0,x_0) + \mathcal{P}_\epsilon(g-h_n)(0,x_0)| \\
\leq \;& |\mathcal{P}_t(g-h_n)(0,x_0)| + |\,(\mathcal{P}_t - \mathcal{P}_\epsilon)\,h_n(0,x_0)| + |\mathcal{P}_\epsilon(g-h_n)(0,x_0)| \\
\leq \;& 2\,\|g-h_n\| + |\,(\mathcal{P}_t - \mathcal{P}_\epsilon)\,h_n(0,x_0)|
\end{aligned}$$

Using the right-continuity of $t \mapsto \mathcal{P}_t h_n(0,x_0)$ for any $n \geq 0$, taking $t \downarrow \epsilon$ then $n \to \infty$, yields

$$\lim_{t\downarrow\epsilon} \mathcal{P}_t g(0,x_0) = \mathcal{P}_\epsilon g(0,x_0).$$

Thus the two right-continuous functions $t \mapsto \mathcal{P}_t g(0,x_0)$ and $t \mapsto \mathcal{U}_t g(0,x_0)$ have the same Laplace transform by (2.20), which implies they are equal:

$$\forall g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^d), \quad \int g(t,y) q_{0,t}(x_0,dy) = \int g(t,y) p_t(x_0,dy). \tag{2.21}$$

By [40, Proposition 4.4, Chapter 3], $\mathcal{C}_0([0,\infty[\times\mathbb{R}^d)$ is convergence determining, hence separating, allowing us to conclude that $p_t(x_0,dy) = q_{0,t}(x_0,dy)$.

□

Remark 2.1. *Assumptions 2.1, 2.2 and 2.3 are sufficient but not necessary for the well-posedness of the martingale problem. For example, the boundedness Assumption 2.1 may be relaxed to local boundedness, using localization techniques developed in [88, 90]. Such extensions are not trivial and, in the unbounded case, additional conditions are needed to ensure that X does not explode (see [90, Chapter 10]).*

2.2.3 Markovian projection of a semimartingale

The following assumptions on the local characteristics of the semimartingale ξ are almost-sure analogs of Assumptions 2.1, 2.2 and 2.3:

Assumption 2.4. *β, δ are bounded on $[0,T]$:*

$$\exists K_1 > 0, \forall t \in [0,T], \;\; \|\beta_t\| \leq K_1, \;\; \|\delta_t\| \leq K_1 \qquad \text{a.s.}$$

Assumption 2.5. *The jump compensator μ has a density $m(\omega, t, dy)$ with respect to the Lebesgue measure on $[0,T]$ which satisfies*

$$(i) \qquad \exists K_2 > 0, \forall t \in [0,T], \quad \int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \, m(.,t,dy) \leq K_2 < \infty \qquad \text{a.s.}$$

$$(ii) \quad \text{and} \quad \lim_{R\to\infty} \int_0^T m\left(.,t,\{\|y\| \geq R\}\right) \, dt = 0 \quad \text{a.s.}$$

Assumption 2.6 (Local non-degeneracy)**.**

Either (i) $\exists \epsilon > 0, \forall t \in [0,T[\; {}^t\delta_t \delta_t \geq \epsilon \; I_d$ a.s.

or (ii) $\delta \equiv 0$ and there exists $\beta \in]0,2[, C, K_3 > 0$, and a family $m^\beta(t,dy)$ of positive measures such that

$$\forall t \in [0,T[\qquad m(t,dy) = m^\beta(t,dy) + 1_{\{\|y\|\leq 1\}} \frac{C}{\|y\|^{d+\beta}} \, dy \text{ a.s.},$$

$$\int \left(1 \wedge \|y\|^\beta\right) \, m^\beta(t,dy) \leq K_3, \quad \text{and} \quad \lim_{\epsilon\to 0} \int_{\|y\|\leq\epsilon} \|y\|^\beta \, m^\beta(t,dy) = 0 \text{ a.s.}$$

Note that Assumption 2.5 is only slightly stronger than stating that m is a Lévy kernel since in that case we already have $\int \left(1 \wedge \|y\|^2\right) \, m(.,t,dy) < \infty$. Assumption 2.6 extends the "ellipticity" assumption to the case of pure-jump semimartingales and holds for a large class of semimartingales driven by stable or tempered stable processes.

Theorem 2.2 (Markovian projection). *Assume there exists measurable functions* $a : [0,T] \times \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$, $b : [0,T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$ *and* $n : [0,T] \times \mathbb{R}^d \mapsto \mathcal{R}(\mathbb{R}^d - \{0\})$ *satisfying Assumption 2.2 such that for all* $t \in [0,T]$ *and* $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$,

$$\begin{aligned} &\mathbb{E}\left[\beta_t | \xi_{t^-}\right] = b(t, \xi_{t-}) \quad \text{a.s,} \\ &\mathbb{E}\left[{}^t\delta_t \delta_t | \xi_{t^-}\right] = a(t, \xi_{t-}) \quad \text{a.s,} \\ &\mathbb{E}\left[m(., t, B) | \xi_{t^-}\right] = n(t, B, \xi_{t-}) \quad \text{a.s.} \end{aligned} \tag{2.22}$$

If (β, δ, m) *satisfies Assumptions 2.4, 2.5, 2.6, then there exists a Markov process* $((X_t)_{t\in[0,T]}, \mathbb{Q}_{\xi_0})$, *with infinitesimal generator* $\mathcal{L}$ *defined by* (2.3), *whose marginal distributions mimick those of* ξ:

$$\forall t \in [0,T], \quad X_t \overset{\text{d}}{=} \xi_t.$$

X *is the weak solution of the stochastic differential equation*

$$\begin{aligned} X_t = \xi_0 &+ \int_0^t b(u, X_u)\, du + \int_0^t \Sigma(u, X_u)\, dB_u \\ &+ \int_0^t \int_{\|y\|\le 1} y\, \tilde{N}(du\ dy) + \int_0^t \int_{\|y\|>1} y\, N(du\ dy), \end{aligned} \tag{2.23}$$

where (B_t) *is an* n*-dimensional Brownian motion,* N *is an integer-valued random measure on* $[0,T] \times \mathbb{R}^d$ *with compensator* $n(t, dy, X_{t-})\, dt$, $\tilde{N} = N - n$ *the associated compensated random measure and* $\Sigma \in C^0([0,T] \times \mathbb{R}^d, M_{d\times n}(\mathbb{R}))$ *such that* ${}^t\Sigma(t,z)\Sigma(t,z) = a(t,z)$.

We will call $(X, \mathbb{Q}_{\xi_0})$ the *Markovian projection* of ξ.

Proof. First, we observe that n is a Lévy kernel : for any $(t,z) \in [0,T] \times \mathbb{R}^d$

$$\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right)\, n(t, dy, z) = \mathbb{E}\left[\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right)\, m(t, dy) | \xi_{t^-} = z\right] < \infty \text{ a.s.,}$$

using Fubini's theorem and Assumption 2.5. Consider now the case of a pure jump semimartingale verifying (ii) and define, for $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$,

$$\forall z \in \mathbb{R}^d \quad n^\beta(t, B, z) = \mathbb{E}\left[\int_B m(t, dy, \omega) - \frac{C\, dy}{\|y\|^{d+\beta}} | \xi_{t^-} = z\right].$$

As argued above, n^β is a Lévy kernel on $\mathbb{R}^d$. Assumptions 2.4 and 2.5 imply that (b,a,n) satisfies Assumption 2.1. Furthermore, under assumptions either (i) or (ii) for (δ,m), Assumption 2.3 holds for (b,a,n). Together with Assumption 2.2 yields that $\mathcal{L}$ is a non-degenerate operator and Proposition 2.1 implies that the martingale problem for $(\mathcal{L}_t)_{t\in[0,T]}$ on the domain $\mathcal{C}_0^\infty(\mathbb{R}^d)$ is well-posed. Denote $((X_t)_{t\in[0,T]},\mathbb{Q}_{\xi_0})$ its unique solution starting from ξ_0 and $q_t(\xi_0,dy)$ the marginal distribution of X_t.

Let $f\in\mathcal{C}_0^\infty(\mathbb{R}^d)$. Itô's formula yields

$$
\begin{aligned}
f(\xi_t) &= f(\xi_0)+\sum_{i=1}^d\int_0^t\sum_{i=1}^d\frac{\partial f}{\partial x_i}(\xi_{s^-})\,d\xi_s^i+\frac{1}{2}\int_0^t\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,{}^t\delta_s\delta_s\right]\,ds\\
&+\sum_{s\le t}\left[f(\xi_{s^-}+\Delta\xi_s)-f(\xi_{s^-})-\sum_{i=1}^d\frac{\partial f}{\partial x_i}(\xi_{s^-})\Delta\xi_s^i\right]\\
&= f(\xi_0)+\int_0^t\nabla f(\xi_{s^-}).\beta_s\,ds+\int_0^t\nabla f(\xi_{s^-}).\delta_s dW_s\\
&+\frac{1}{2}\int_0^t\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,{}^t\delta_s\delta_s\right]\;ds+\int_0^t\int_{\|y\|\le 1}\nabla f(\xi_{s^-}).y\,\tilde{M}(ds\,dy)\\
&+\int_0^t\int_{\mathbb{R}^d}\left(f(\xi_{s^-}+y)-f(\xi_{s^-})-1_{\{\|y\|\le 1\}}\,y.\nabla f(\xi_{s^-})\right)\,M(ds\,dy).
\end{aligned}
$$

We note that

- since $\|\nabla f\|$ is bounded $\int_0^t\int_{\|y\|\le 1}\nabla f(\xi_{s^-}).y\,\tilde{M}(ds\,dy)$ is a square-integrable martingale.
- $\int_0^t\int_{\|y\|>1}\nabla f(\xi_{s^-}).y\,M(ds\,dy)<\infty$ a.s. since $\|\nabla f\|$ is bounded.
- since $\nabla f(\xi_{s^-})$ and δ_s are uniformly bounded on $[0,T]$, $\int_0^t\nabla f(\xi_{s^-}).\delta_s dW_s$ is a martingale.

Hence, taking expectations, we obtain:

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] &= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] + \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})^t\delta_s\delta_s\right]\, ds\right] \\
&+ \mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, M(ds\, dy)\right] \\
&= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] + \mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,{}^t\delta_s\delta_s\right]\, ds\right] \\
&+ \mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, m(s,dy)\, ds\right].
\end{aligned}$$

Observing that:

$$\begin{aligned}
&\mathbb{E}^{\mathbb{P}}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] \le \|\nabla f\|\, \mathbb{E}^{\mathbb{P}}\left[\int_0^t \|\beta_s\|\, ds\right] < \infty, \\
&\mathbb{E}^{\mathbb{P}}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,{}^t\delta_s\delta_s\right]\right] \le \|\nabla^2 f\|\, \mathbb{E}^{\mathbb{P}}\left[\int_0^t \|\delta_s\|^2\, ds\right] < \infty, \\
&\mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\mathbb{R}^d} \left\|f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right\|\, m(s,dy)\, ds\right] \\
&\quad \le \frac{\|\nabla^2 f\|}{2}\mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\|y\|\le 1} \|y\|^2\, m(s,dy)\, ds\right] + 2\|f\|\mathbb{E}^{\mathbb{P}}\left[\int_0^t \int_{\|y\|>1} m(s,dy)\, ds\right] < +\infty,
\end{aligned}$$

we may apply Fubini's theorem to obtain

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] &= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \int_0^t \mathbb{E}^{\mathbb{P}}\left[\nabla f(\xi_{s^-}).\beta_s\right]\, ds + \frac{1}{2}\int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,{}^t\delta_s\delta_s\right]\right]\, ds \\
&+ \int_0^t \mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\le 1\}}\, y.\nabla f(\xi_{s^-})\right)\, m(s,dy)\right]\, ds.
\end{aligned}$$

Conditioning on ξ_{t-} and using the iterated expectation property,

$$\begin{aligned}
\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] &= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \int_0^t \mathbb{E}^{\mathbb{P}}\left[\nabla f(\xi_{s^-}).\mathbb{E}^{\mathbb{P}}\left[\beta_s|\xi_{s-}\right]\right]\,ds \\
&+ \frac{1}{2}\int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,\mathbb{E}^{\mathbb{P}}\left[{}^t\delta_s\delta_s|\xi_{s-}\right]\right]\right]\,ds \\
&+ \int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathbb{E}^{\mathbb{P}}\left[\int_{\mathbb{R}^d}\left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\leq 1\}}\,y.\nabla f(\xi_{s^-})\right)\,m(s,dy)|\xi_{s-}\right]\right]\,ds \\
&= \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \int_0^t \mathbb{E}^{\mathbb{P}}\left[\nabla f(\xi_{s^-}).b(s,\xi_{s-})\right]\,ds + \frac{1}{2}\int_0^t \mathbb{E}^{\mathbb{P}}\left[\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,a(s,\xi_{s-})\right]\right]\,ds \\
&+ \int_0^t\int_{\mathbb{R}^d} \mathbb{E}^{\mathbb{P}}\left[\left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - 1_{\{\|y\|\leq 1\}}\,y.\nabla f(\xi_{s^-})\right)\,n(s,dy,\xi_{s-})\right]\,ds.
\end{aligned}$$

Hence

$$\mathbb{E}^{\mathbb{P}}\left[f(\xi_t)\right] = \mathbb{E}^{\mathbb{P}}\left[f(\xi_0)\right] + \mathbb{E}^{\mathbb{P}}\left[\int_0^t \mathcal{L}_s f(\xi_{s-})\,ds\right]. \tag{2.24}$$

Let $p_t(dy)$ denote the law of (ξ_t) under $\mathbb{P}$, (2.24) writes:

$$\int_{\mathbb{R}^d} p_t(dy)f(y) = \int_{\mathbb{R}^d} p_0(dy)f(y) + \int_0^t\int_{\mathbb{R}^d} p_s(dy)\mathcal{L}_s f(y)\,ds. \tag{2.25}$$

Hence $p_t(dy)$ satisfies the Kolmogorov forward equation (2.7) for the operator $\mathcal{L}$ with the initial condition $p_0(dy) = \mu_0(dy)$ where μ_0 denotes the law of ξ_0. Applying Theorem 2.1, the flows $q_t(\xi_0, dy)$ of X_t and $p_t(dy)$ of ξ_t are the same on $[0,T]$. This ends the proof.

□

Remark 2.2 (Mimicking conditional distributions)**.** *The construction in Theorem 2.2 may also be carried out using*

$$\mathbb{E}\left[\beta_t|\xi_{t^-},\,\mathcal{F}_0\right] = b_0(t,\xi_{t-})\text{ a.s},$$
$$\mathbb{E}\left[{}^t\delta_t\delta_t|\xi_{t^-},\,\mathcal{F}_0\right] = a_0(t,\xi_{t-})\text{ a.s},$$
$$\mathbb{E}\left[m(.,t,B)|\xi_{t^-},\,\mathcal{F}_0\right] = n_0(t,B,\xi_{t-})\text{ a.s},$$

instead of (b,a,n) *in* (2.36)*. If* (b_0,a_0,n_0) *satisfies Assumption* (2.3)*, then following the same procedure we can construct a Markov process* $(X, \mathbb{Q}^0_{\xi_0})$ *whose infinitesimal generator has coefficients* (b_0,a_0,n_0) *such that*

$$\forall f\in\mathcal{C}^0_b(\mathbb{R}^d), \forall t\in[0,T]\quad \mathbb{E}^{\mathbb{P}}\left[f(\xi_t)|\mathcal{F}_0\right] = \mathbb{E}^{\mathbb{Q}^0_{\xi_0}}\left[f(X_t)\right],$$

i.e. the marginal distribution of X_t matches the conditional distribution of ξ_t given $\mathcal{F}_0$.

Remark 2.3. *For Ito processes (i.e. continuous semimartingales of the form* (2.2) *with $\mu = 0$), Gyöngy [51, Theorem 4.6] gives a "mimicking theorem" under the non-degeneracy condition ${}^t\delta_t.\delta_t \geq \epsilon I_d$ which corresponds to our Assumption 2.6, but without requiring the continuity condition (Assumption 2.2) on (b, a, n). Brunick & Shreve [23] extend this result by relaxing the ellipticity condition of [51]. In both cases, the mimicking process X is constructed as a weak solution to the SDE* (2.23) *(without the jump term), but this weak solution does not in general have the Markov property: indeed, it need not even be unique under the assumptions used in [51, 23]. In particular, in the setting used in [51, 23], the law of X is not uniquely determined by its 'infinitesimal generator' $\mathcal{L}$. This makes it difficult to 'compute' quantities involving X, either through simulation or by solving a partial differential equation.*

By contrast, under the additional continuity condition 2.2 on the projected coefficients, X is a Markov process whose law is uniquely determined by its infinitesimal generator $\mathcal{L}$ and whose marginals are the unique solution of the Kolmogorov forward equation (2.7)*. This makes it possible to compute the marginals of X by simulating the SDE* (2.23) *or by solving a forward PIDE.*

It remains to be seen whether the additional Assumption 2.2 is verified in most examples of interest. We will show in Section 2.4 that this is indeed the case.

Remark 2.4 (Markovian projection of a Markov process)**.** *The term* Markovian projection *is justified by the following remark: if the semimartingale ξ is already a Markov process and satisfies the assumption of Theorem 2.2, then the uniqueness in law of the solution to the martingale problem for $\mathcal{L}$ implies that the Markovian projection $(X, \mathbb{Q}_{\xi_0})$ of ξ has the same law as $(\xi, \mathbb{P}_{\xi_0})$. So the map which associates (the law $\mathbb{Q}_{\xi_0}$ of) X to ξ is an involution and may be viewed as a projection of $\mathbb{P}$ on the set of Markov processes.*

This property contrasts with other constructions of mimicking processes [7, 28, 51, 52, 74] which fail to be involutive. A striking example is the construction, by Hamza & Klebaner [52], of discontinuous martingales whose marginals match those of a Gaussian Markov process.

2.2.4 Forward equations for semimartingales

Theorem 2.1 and Theorem 2.2 allow us to obtain a forward PIDE which extends the Kolmogorov forward equation to semimartingales which verify the Assumptions of Theorem 2.2:

Theorem 2.3. *Let ξ be a semimartingale given by (2.2) satisfying the assumptions of Theorem 2.2. Denote $p_t(dx)$ the law of ξ_t on $\mathbb{R}^d$. Then $(p_t)_{t\in[0,T]}$ is the unique solution, in the sense of distributions, of the forward equation*

$$\forall t \in [0,T], \quad \frac{\partial p_t}{\partial t} = \mathcal{L}_t^{\star} . \, p_t, \tag{2.26}$$

with initial condition $p_0 = \mu_0$, where μ_0 denotes the law of ξ_0,
where $\mathcal{L}^{\star}$ is the adjoint of $\mathcal{L}$, defined by

$$\begin{aligned} \forall g &\in C_0^{\infty}(\mathbb{R}^d, \mathbb{R}), \\ \mathcal{L}_t^{\star} g(x) &= -\nabla\left[b(t,x)g(x)\right] + \nabla^2\left[\frac{a(t,x)}{2}g(x)\right] \\ &+ \int_{\mathbb{R}^d}\left[g(x-z)n(t,z,x-z) - g(x)n(t,z,x) - 1_{\|z\|\leq 1} z.\nabla\left[g(x)n(t,dz,x)\right]\right], \end{aligned} \tag{2.27}$$

where the coefficients b, a, n are defined as in (2.36)*.*

Proof. The existence and uniqueness is a direct consequence of Theorem 2.1 and Theorem 2.2. To finish the proof, let compute $\mathcal{L}_t^{\star}$. Viewing p_t as an element of the dual of $C_0^{\infty}(\mathbb{R}^d)$, (2.7) rewrites : for $f \in C_0^{\infty}(\mathbb{R}^d, \mathbb{R})$

$$\forall f \in \mathcal{C}_0^{\infty}(\mathbb{R}^d, \mathbb{R}), \qquad \int f(y)\frac{dp}{dt}(dy) = \int p_t(dy)\mathcal{L}_t f(y).$$

We have

$$\forall f \in C_0^{\infty}(\mathbb{R}^d), \forall t \leq t' < T \quad < \frac{p_{t'} - p_t}{t' - t}, f > \overset{t' \to t}{\to} < p_t, \mathcal{L}_t f > = < \mathcal{L}_t^{*} p_t, f >,$$

where $< .,. >$ is the duality product.

For $z \in \mathbb{R}^d$, define the translation operator τ^z by $\tau_z f(x) = f(x+z)$. Then

$$\begin{aligned}
&\int p_t(dx)\, \mathcal{L}_t f(x) \\
= \quad &\int p_t(dx) \Big[b(t,x)\nabla f(x) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(x)\, a(t,x)\right] \\
&\qquad + \int_{|z|>1} (\tau_z f(x) - f(x)) n(t,dz,x) \\
&\qquad + \int_{|z|\le 1} (\tau_z f(x) - f(x) - z.\nabla f(x))\, n(t,dz,x) \Big] \\
= \quad &\int \Big[-f(x)\frac{\partial}{\partial x}[b(t,x)p_t(dx)] + f(x)\frac{\partial^2}{\partial x^2}[\frac{a(t,x)}{2} p_t(dx)] \\
&\qquad + \int_{|z|>1} f(x)(\tau_{-z}(p_t(dx)n(t,dz,x)) - p_t(dx)n(t,dz,x)) \\
&\qquad + \int_{|z|\le 1} f(x)(\tau_{-z}(p_t(dx)n(t,dz,x)) - p_t(dx)n(t,dz,x)) \\
&\qquad - z\frac{\partial}{\partial x}(p_t(dx)n(t,dz,x)) \Big],
\end{aligned}$$

allowing to identify $\mathcal{L}^\star$. □

2.2.5 Martingale-preserving property

An important property of the construction of ξ in Theorem 2.2 is that it preserves the (local) martingale property: if ξ is a local martingale, so is X:

Proposition 2.2 (Martingale preserving property)**.**

1. *If ξ is a* local martingale *which satisfies the assumptions of Theorem 2.2, then its Markovian projection $(X_t)_{t\in[0,T]}$ is a local martingale on $(\Omega_0, \mathcal{B}_t, \mathbb{Q}_{\xi_0})$.*
2. *If furthermore*

$$\mathbb{E}^{\mathbb{P}}\left[\int_0^T \int_{\mathbb{R}^d} \|y\|^2 m(t,dy)dt\right] < \infty,$$

then $(X_t)_{t\in[0,T]}$ is a square-integrable martingale.

Proof. 1) If ξ is a local martingale then the uniqueness of its semimartingale decomposition entails that

$$\beta_t + \int_{\|y\|\geq 1} y\, m(t, dy) = 0 \qquad dt \times \mathbb{P} - a.s.$$

hence $$\mathbb{Q}_{\xi_0}\left(\forall t \in [0,T], \quad \int_0^t ds \left[b(s, X_{s-}) + \int_{\|y\|\geq 1} y\, n(s, dy, X_{s-}) \right] = 0 \right) = 1.$$

The assumptions on m, δ then entail that X, as a sum of an Ito integral and a compensated Poisson integral, is a local martingale.

2) If $\mathbb{E}^{\mathbb{P}}\left[\int \|y\|^2 \mu(dt\, dy)\right] < \infty$ then

$$\mathbb{E}^{\mathbb{Q}_{\xi_0}}\left[\int \|y\|^2 n(t, dy, X_{t-})\right] < \infty,$$

and the compensated Poisson integral in X is a square-integrable martingale.

□

2.3 Mimicking a semimartingale driven by a Poisson random measure

The representation (2.2) is not the most commonly used in applications, where a process is constructed as the solution to a stochastic differential equation driven by a Brownian motion and a Poisson random measure

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int \psi_s(y)\, \tilde{N}(ds\, dy), \tag{2.28}$$

where $\xi_0 \in \mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, β and δ are non-anticipative càdlàg processes, N is a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with intensity $\nu(dy)\, dt$ where

$$\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu(dy) < \infty, \qquad \tilde{N} = N - \nu(dy)dt, \tag{2.29}$$

and the random jump amplitude $\psi : [0,T] \times \Omega \times \mathbb{R}^d \mapsto \mathbb{R}^d$ is $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable, where $\mathcal{P}$ is the predictable σ-algebra on $[0,T]\times\Omega$. In this section, we shall assume that

$$\forall t \in [0,T], \quad \psi_t(\omega, 0) = 0 \quad \text{and} \quad \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \left(1 \wedge \|\psi_s(., y)\|^2\right) \nu(dy)\, ds\right] < \infty.$$

The difference between this representation and (2.2) is the presence of a random jump amplitude $\psi_t(\omega, .)$ in (2.28). The relation between these two representations for semimartingales has been discussed in great generality in [39, 63]. Here we give a less general result which suffices for our purpose. The following result expresses ζ in the form (2.2) suitable for applying Theorem 2.2.

Lemma 2.1 (Absorbing the jump amplitude in the compensator)**.**

$$\zeta_t = \zeta_0 + \int_0^t \beta_s \, ds + \int_0^t \delta_s \, dW_s + \int_0^t \int \ \psi_s(z) \, \tilde{N}(ds \, dz)$$

can be also represented as

$$\zeta_t = \zeta_0 + \int_0^t \beta_s \, ds + \int_0^t \delta_s \, dW_s + \int_0^t \int y \, \tilde{M}(ds \, dy), \tag{2.30}$$

where M is an integer-valued random measure on $[0, T] \times \mathbb{R}^d$ with compensator $\mu(\omega, dt, dy)$ given by

$$\forall A \in \mathcal{B}(\mathbb{R}^d - \{0\}), \qquad \mu(\omega, dt, A) = \nu(\psi_t^{-1}(\omega, A)) \, dt,$$

where $\psi_t^{-1}(\omega, A) = \{z \in \mathbb{R}^d, \psi_t(\omega, z) \in A\}$ denotes the inverse image of A under the partial map ψ_t.

Proof. The result can be deduced from [39, Théorème 12] but we sketch here the proof for completeness. A Poisson random measure N on $[0, T] \times \mathbb{R}^d$ can be represented as a counting measure for some random sequence (T_n, U_n) with values in $[0, T] \times \mathbb{R}^d$

$$N = \sum_{n \geq 1} 1_{\{T_n, U_n\}}. \tag{2.31}$$

Let M be the integer-valued random measure defined by:

$$M = \sum_{n \geq 1} 1_{\{T_n, \psi_{T_n}(., U_n)\}}. \tag{2.32}$$

μ, the *predictable* compensator of M is characterized by the following property [61, Thm 1.8.]: for any positive $\mathcal{P} \otimes \mathcal{B}(\mathbb{R}^d)$-measurable map $\chi : [0, T] \times \Omega \times \mathbb{R}^d \to \mathbb{R}^+$ and any $A \in \mathcal{B}(\mathbb{R}^d - \{0\})$,

$$\mathbb{E}\left[\int_0^t \int_A \chi(s, y) \, M(ds \, dy)\right] = \mathbb{E}\left[\int_0^t \int_A \chi(s, y) \, \mu(ds \, dy)\right]. \tag{2.33}$$

Similarly, for $B \in \mathcal{B}(\mathbb{R}^d - \{0\})$

$$\mathbb{E}\left[\int_0^t \int_B \chi(s,y)\, N(ds\, dy)\right] = \mathbb{E}\left[\int_0^t \int_B \chi(s,y)\, \nu(dy)\, ds\right].$$

Using formulae (2.31) and (2.32):

$$\begin{aligned}
\mathbb{E}\left[\int_0^t \int_A \chi(s,y)\, M(ds\, dy)\right] &= \mathbb{E}\left[\sum_{n\geq 1} \chi(T_n, \psi_{T_n}(.,U_n))\right] \\
&= \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(.,z))\, N(ds\, dz)\right] \\
&= \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(.,z))\, \nu(dz)\, ds\right]
\end{aligned}$$

Formula (2.33) and the above equalities lead to:

$$\mathbb{E}\left[\int_0^t \int_A \chi(s,y)\, \mu(ds\, dy)\right] = \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(.,z))\, \nu(dz)\, ds\right].$$

Since ψ is a predictable random function, the uniqueness of the predictable compensator μ (take $\phi \equiv Id$ in [61, Thm 1.8.]) entails

$$\mu(\omega, dt, A) = \nu(\psi_t^{-1}(\omega, A))\, dt. \tag{2.34}$$

Formula (2.34) defines a random measure μ which is a Lévy kernel

$$\int_0^t \int \left(1 \wedge \|y\|^2\right)\, \mu(dy\, ds) = \int_0^t \int \left(1 \wedge \|\psi_s(.,y)\|^2\right)\, \nu(dy)\, ds < \infty.$$

□

In the case where $\psi_t(\omega,.) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is invertible and differentiable, we can characterize the density of the compensator μ as follows:

Lemma 2.2 (Differentiable case). *If the Lévy measure $\nu(dz)$ has a density $\nu(z)$ and if $\psi_t(\omega,.) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is a $\mathcal{C}^1(\mathbb{R}^d, \mathbb{R}^d)$-diffeomorphism, then ζ, given in* (2.28)*, has the representation*

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int y\, \tilde{M}(ds\, dy),$$

where M is an integer-valued random measure with compensator

$$\mu(\omega; dt, dy) = 1_{\psi_t(\omega,\mathbb{R}^d)}(y)\, |\det\nabla_y\psi_t|^{-1}\, (\omega, \psi_t^{-1}(\omega, y))\, \nu(\psi_t^{-1}(\omega, y))\, dt\, dy,$$

where $\nabla_y\psi_t$ denotes the Jacobian matrix of $\psi_t(\omega, .)$.

Proof. We recall from the proof of Lemma 2.1:

$$\mathbb{E}\left[\int_0^t \int_A \chi(s, y)\, \mu(ds\, dy)\right] = \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(., z))\, \nu(z)\, ds\, dz\right].$$

Proceeding to the change of variable $\psi_s(., z) = y$:

$$\begin{aligned} & \mathbb{E}\left[\int_0^t \int_{\psi_s^{-1}(.,A)} \chi(s, \psi_s(., z))\, \nu(z)\, ds\, dz\right] \\ = \ & \mathbb{E}\left[\int_0^t \int_A 1_{\{\psi_s(\mathbb{R}^d)\}}(y)\, \chi(s, y)\ |\det\nabla\psi_s|^{-1}\, (., \psi_s^{-1}(., y))\, \nu(\psi_s^{-1}(., y)) ds\, dy\right]. \end{aligned}$$

The density appearing in the right hand side is predictable since ψ is a predictable random function. The uniqueness of the predictable compensator μ yields the result. □

Let us combine Lemma 2.2 and Theorem 2.2. To proceed, we make a further assumption.

Assumption 2.7. *The Lévy measure ν admits a density $\nu(y)$ with respect to the Lebesgue measure on $\mathbb{R}^d$ and:*

$$\forall t \in [0, T]\, \exists K_2 > 0 \quad \int_0^t \int_{\|y\|>1} \left(1 \wedge \|\psi_s(., y)\|^2\right)\, \nu(y)\, dy\, ds < K_2 \text{ a.s.}$$

and

$$\lim_{R\to\infty} \int_0^T \nu\left(\psi_t^{-1}\left(\{\|y\| \geq R\}\right)\right)\, dt = 0 \quad \text{a.s.}$$

Theorem 2.4. *Let (ζ_t) be an Ito semimartingale defined on $[0, T]$ by the given the decomposition*

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int \psi_s(y)\, \tilde{N}(ds\, dy),$$

where $\psi_t(\omega,.) : \mathbb{R}^d \mapsto \mathbb{R}^d$ is invertible and differentiable with inverse $\phi_t(\omega,.)$. Define

$$m(t,y) = 1_{\{y\in\psi_t(\mathbb{R}^d)\}}\,|\mathrm{det}\nabla\psi_t|^{-1}\,(\psi_t^{-1}(y))\,\nu(\psi_t^{-1}(y)). \tag{2.35}$$

Assume there exists measurable functions $a : [0,T]\times\mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R}), b : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$ and $j : (t,x) \in [0,T]\times\mathbb{R}^d) \to \mathcal{R}(\mathbb{R}^d - \{0\})$ satisfying Assumption 2.2 such that for $(t,z)\in[0,T]\times\mathbb{R}^d, B\in\mathcal{B}(\mathbb{R}^d-\{0\})$,

$$\begin{aligned}\mathbb{E}\left[\beta_t|\zeta_{t^-}\right] &= b(t,\zeta_{t-})\text{ a.s,}\\ \mathbb{E}\left[{}^t\delta_t\delta_t|\zeta_{t^-}\right] &= a(t,\zeta_{t-})\text{ a.s,}\\ \mathbb{E}\left[m(.,t,B)|\zeta_{t^-}\right] &= j(t,B,\zeta_{t-})\text{ a.s.}\end{aligned} \tag{2.36}$$

If β and δ satisfy Assumption 2.4, ν Assumption 2.7, (δ,m) satisfy Assumptions 2.5-2.6, then the stochastic differential equation

$$X_t = \zeta_0 + \int_0^t b(u,X_u)\,du + \int_0^t \Sigma(u,X_u)\,dB_u + \int_0^t\int y\,\tilde{J}(du\;dy), \tag{2.37}$$

where (B_t) is an n-dimensional Brownian motion, J is an integer valued random measure on $[0,T]\times\mathbb{R}^d$ with compensator $j(t,dy,X_{t-})\,dt$, $\tilde{J} = J - j$ and $\Sigma : [0,T]\times\mathbb{R}^d \mapsto M_{d\times n}(\mathbb{R})$ is a continuous function such that ${}^t\Sigma(t,z)\Sigma(t,z) = a(t,z)$, admits a unique weak solution $((X_t)_{t\in[0,T]},\mathbb{Q}_{\zeta_0})$ whose marginal distributions mimick those of ζ:

$$\forall t\in[0,T]\quad X_t \overset{\mathrm{d}}{=} \zeta_t.$$

Under $\mathbb{Q}_{\zeta_0}$, X is a Markov process with infinitesimal generator $\mathcal{L}$ given by (2.3).

Proof. We first use Lemma 2.2 to obtain the representation (2.30) of ζ:

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\,ds + \int_0^t \delta_s\,dW_s + \int_0^t\int y\,\tilde{M}(ds\,dy)$$

Then, we observe that

$$\begin{aligned}&\int_0^t\int y\,\tilde{M}(ds\,dy)\\ =\;&\int_0^t\int_{\|y\|\le 1} y\,\tilde{M}(ds\,dy) + \int_0^t\int_{\|y\|>1} y\left[M(ds\,dy) - m(s,dy)\,ds\right]\\ =\;&\int_0^t\int_{\|y\|\le 1} y\,\tilde{M}(ds\,dy) + \int_0^t\int_{\|y\|>1} y\,M(ds\,dy) - \int_0^t\int_{\|y\|>1} y\,m(s,dy)\,ds,\end{aligned}$$

where the terms above are well-defined thanks to Assumption 2.7. Lemma 2.2 leads to:

$$\int_0^t \int_{\|y\|>1} y\, m(s,dy)\, ds = \int_0^t \int_{\|\psi_s(y)\|>1} \|\psi_s(.,y)\|\, \nu(y)\, dy\, ds.$$

Hence:

$$\zeta_t = \zeta_0 + \left[\int_0^t \beta_s\, ds - \int_0^t \int_{\|\psi_s(y)\|>1} \|\psi_s(.,y)\|\, \nu(y)\, dy\, ds\right] + \int_0^t \delta_s\, dW_s$$
$$+ \int_0^t \int_{\|y\|\leq 1} y\, \tilde{M}(ds\; dy) + \int_0^t \int_{\|y\|>1} y\, M(ds\; dy).$$

This representation has the form (2.2) and Assumptions 2.4 and 2.7 guarantee that the local characteristics of ζ satisfy the assumptions of Theorem 2.2. Applying Theorem 2.2 yields the result. $\square$

2.4 Examples

We now give some examples of stochastic models used in applications, where Markovian projections can be characterized in a more explicit manner than in the general results above. These examples also serve to illustrate that the continuity assumption (Assumption 2.2) on the projected coefficients (b, a, n) in (2.36) can be verified in many useful settings.

2.4.1 Semimartingales driven by a Markov process

In many examples in stochastic modeling, a quantity Z is expressed as a smooth function $f : \mathbb{R}^d \to \mathbb{R}$ of a d-dimensional Markov process Z:

$$\xi_t = f(Z_t) \qquad \text{with} \quad f : \mathbb{R}^d \to \mathbb{R}$$

We will show that in this situation our assumptions will hold for ξ as soon as Z has an infinitesimal generator whose coefficients satisfy Assumptions 2.1, 2.2 and 2.3, allowing us to construct the Markovian Projection of ξ_t.

Consider a time-dependent integro-differential operator $L = (L_t)_{t\in[0,T]}$ defined, for $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$, by

$$\begin{aligned} L_t g(z) &= b_Z(t,z).\nabla g(z) + \sum_{i,j=1}^{d} \frac{(a_Z)_{ij}(t,x)}{2} \frac{\partial^2 g}{\partial x_i \partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d} [g(z+\psi_Z(t,z,y)) - g(z) - \psi_Z(t,y,z).\nabla g(z)] \nu_Z(y) dy, \end{aligned} \tag{2.38}$$

where $b_Z : [0,T]\times\mathbb{R}^d \mapsto \mathbb{R}^d$, $a_Z : [0,T]\times\mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$, and $\psi_Z : [0,T]\times\mathbb{R}^d\times\mathbb{R}^d$ are measurable functions and ν_Z is a Lévy density.
If one assume that

$$\begin{aligned} &\psi_Z(.,.,0) = 0 \quad \psi_Z(t,z,.)\,\text{is a}\,\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d) - \text{diffeomorphism}, \\ &\forall t\in[0,T]\,\forall z\in\mathbb{R}^d\,\mathbb{E}\left[\int_0^t \int_{\{\|y\|\ge 1\}} \left(1 \wedge \|\psi_Z(s,z,y)\|^2\right)\,\nu_Z(y)\,dy\,ds\right] < \infty, \end{aligned} \tag{2.39}$$

then applying Lemma 2.2, (2.38) rewrites, for $g \in \mathcal{C}_0^\infty(\mathbb{R}^d)$:

$$\begin{aligned} L_t g(x) &= b_Z(t,x).\nabla g(x) + \sum_{i,j=1}^{d} \frac{(a_Z)_{ij}(t,x)}{2} \frac{\partial^2 g}{\partial x_i \partial x_j}(x) \\ &\quad + \int_{\mathbb{R}^d} [g(x+y) - g(x) - \, y.\nabla g(x)] m_Z(t,y,x)\,dy, \end{aligned} \tag{2.40}$$

where

$$m_Z(t,y,x) = 1_{\{y\in\psi_Z(t,\mathbb{R}^d,x)\}}\left|\det\nabla\psi_Z\right|^{-1}(t,x,\psi_Z^{-1}(t,x,y))\,\nu_Z(\psi_Z^{-1}(t,x,y)). \tag{2.41}$$

Throughout this section we shall assume that (b_Z, a_Z, m_Z) satisfy Assumptions 2.1, 2.2 and 2.3. Proposition 2.1 then implies that for any $Z_0 \in \mathbb{R}^d$, the SDE,

$$\begin{aligned} \forall t\in[0,T] \quad Z_t &= Z_0 + \int_0^t b_Z(u,Z_{u-})\,du + \int_0^t a_Z(u,Z_{u-})\,dW_u \\ &\quad + \int_0^t \int \psi_Z(u,Z_{u-},y)\,\tilde{N}(du\;dy), \end{aligned} \tag{2.42}$$

admits a weak solution $((Z_t)_{t\in[0,T]}, \mathbb{Q}_{Z_0})$, unique in law, with (W_t) an n-dimensional Brownian motion, N a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with compensator $\nu_Z(y)\,dy\,dt$, $\tilde{N}$ the associated compensated random measure. Under $\mathbb{Q}_{Z_0}$, Z is a Markov process with infinitesimal generator L.

Consider now the process

$$\xi_t = f(Z_t). \tag{2.43}$$

The aim of this section is to build in an explicit manner the Markovian Projection of ξ_t for a sufficiently large class of functions f.

Let us first rewrite ξ_t in the form (2.2):

Proposition 2.3. *Let $f \in \mathcal{C}^2(\mathbb{R}^d, \mathbb{R})$ with bounded derivatives such that,*

$$\forall (z_1, \cdots, z_{d-1}) \in \mathbb{R}^{d-1}, \quad u \mapsto f(z_1, \ldots, z_{d-1}, u) \text{ is a } \mathcal{C}^1(\mathbb{R},\mathbb{R})-\text{diffeomorphism.}$$

Assume that (a_Z, m_Z) satisfy Assumption 2.3, then $\xi_t = f(Z_t)$ admits the following semimartingale decomposition:

$$\xi_t = \xi_0 + \int_0^t \beta_s\,ds + \int_0^t \delta_s\,dB_s + \int_0^t \int u\,\tilde{K}(ds\,du),$$

where

$$\begin{cases} \beta_t &= \nabla f(Z_{t^-}).b_Z(t, Z_{t-}) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_{t-})^t a_Z(t, Z_{t-}) a_Z(t, Z_{t-})\right] \\ &\quad + \int_{\mathbb{R}^d} \left(f(Z_{t^-} + \psi_Z(t, Z_{t-}, y)) - f(Z_{t^-}) - \psi_Z(t, Z_{t-}, y).\nabla f(Z_{t^-}) \right) \nu_Z(y)\,dy, \\ \delta_t &= \|\nabla f(Z_{t-}) a_Z(t, Z_{t-})\|, \end{cases} \tag{2.44}$$

B is a real valued Brownian motion and K is an integer-valued random measure on $[0, Tj] \times \mathbb{R}$ with compensator $k(t, Z_{t-}, u)\,du\,dt$ defined for all $z \in \mathbb{R}^d$ and for any $u > 0$ (and analogously for $u < 0$) via:

$$k(t, z, [u, \infty[) = \int_{\mathbb{R}^d} 1_{\{f(z+\psi_Z(t,z,y))-f(z)\geq u\}}\,\nu_Z(y)\,dy. \tag{2.45}$$

and $\tilde{K}$ its compensated random measure.

Proof. Applying Itô's formula to $\xi_t = f(Z_t)$ yields

$$\begin{aligned}
\xi_t &= \xi_0 + \int_0^t \nabla f(Z_{s^-}).b_Z(s, Z_{s-})\, ds + \int_0^t \nabla f(Z_{s^-}).a_Z(s, Z_{s-}) dW_s \\
&+ \frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(Z_{s-})^t a_Z(s, Z_{s-}) a_Z(s, Z_{s-})\right]\, ds + \int_0^t \nabla f(Z_{s^-}).\psi_Z(s, Z_{s-}, y)\ \tilde{N}(ds\, dy) \\
&+ \int_0^t \int_{\mathbb{R}^d} \left(f(Z_{s^-} + \psi_Z(s, Z_{s-}, y)) - f(Z_{s^-}) - \psi_Z(s, Z_{s-}, y).\nabla f(Z_{s^-})\right)\ N(ds\, dy) \\
&= \xi_0 + \int_0^t \Big[\nabla f(Z_{s^-}).b_Z(s, Z_{s-}) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_{s-})^t a_Z(s, Z_{s-}) a_Z(s, Z_{s-})\right] \\
&\qquad + \int_{\mathbb{R}^d} \left(f(Z_{s^-} + \psi_Z(s, Z_{s-}, y)) - f(Z_{s^-}) - \psi_Z(s, Z_{s-}, y).\nabla f(Z_{s^-})\right)\ \nu_Z(y)\, dy\Big]\, ds \\
&+ \int_0^t \nabla f(Z_{s^-}).a_Z(s, Z_{s-}) dW_s + \int_0^t \int_{\mathbb{R}^d} \left(f(Z_{s^-} + \psi_Z(s, Z_{s-}, y)) - f(Z_{s^-})\right)\ \tilde{N}(ds\, dy).
\end{aligned}$$

Given Assumption 2.3, either

$$\forall R > 0\, \forall t \in [0, T] \quad \inf_{\|z\| \le R}\ \inf_{x \in \mathbb{R}^d,\, \|x\|=1} {}^t x.a_Z(t, z).x > 0, \tag{2.46}$$

then $(B_t)_{t \in [0,T]}$ defined by

$$dB_t = \frac{\nabla f(Z_{t^-}).a_Z(t, Z_{t-}) W_t}{\|\nabla f(Z_{t-}) a_Z(t, Z_{t-})\|},$$

is a continuous local martingale with $[B]_t = t$ thus a Brownian motion, or $a_Z \equiv 0$, then ξ is a pure-jump semimartingale. Define $\mathcal{K}_t$

$$\mathcal{K}_t = \int_0^t \int \Psi_Z(s, Z_{s-}, y)\, \tilde{N}(ds\, dy),$$

with $\Psi_Z(t, z, y) = \psi_Z(t, z, \kappa_z(y))$ where

$$\begin{aligned}
\kappa_z(y) : \mathbb{R}^d &\to \mathbb{R}^d \\
y &\to (y_1, \cdots, y_{d-1}, f(z + y) - f(z)).
\end{aligned}$$

Since for any $z \in \mathbb{R}^d$, $\left|\det \nabla_y \kappa_z\right|(y) = \left|\frac{\partial}{\partial y_d} f(z + y)\right| > 0$, one can define

$$\kappa_z^{-1}(y) = (y_1, \cdots, y_{d-1}, F_z(y)) \quad F_z(y) : \mathbb{R}^d \to \mathbb{R} \quad f(z+(y_1, \cdots, y_{d-1}, F_z(y))) - f(z) = y_d.$$

Considering ϕ the inverse of ψ that is $\phi(t,\psi_Z(t,z,y),z)=y$, define

$$\Phi(t,z,y)=\phi(t,z,\kappa_z^{-1}(y)).$$

Φ corresponds to the inverse of Ψ_Z and Φ is differentiable on $\mathbb{R}^d$ with image $\mathbb{R}^d$. Now, define

$$\begin{aligned} m(t,z,y) &= |\mathrm{det}\nabla_y \Phi(t,z,y)|\ \nu_Z(\Phi(t,z,y)) \\ &= \left|\mathrm{det}\nabla_y \phi(t,z,\kappa_z^{-1}(y))\right| \left|\frac{\partial f}{\partial y_d}(z+\kappa_z^{-1}(y))\right|^{-1} \nu_Z(\phi(t,z,\kappa_z^{-1}(y))). \end{aligned}$$

One observes that

$$\begin{aligned} &\int_0^t \int_{\|y\|>1} \left(1\wedge \|\Psi_Z(s,z,y)\|^2\right) \nu_Z(y)\,dy\,ds \ = \\ &\int_0^t \int_{\|y\|>1} \left(1\wedge \left(\psi^1(s,z,y)^2+\cdots+\psi^{d-1}(s,z,y)^2+(f(z+\psi_Z(s,z,y))-f(z))^2\right)\right) \nu_Z(y)\,dy\,ds \\ \le\ &\int_0^t \int_{\|y\|>1} \left(1\wedge \left(\psi^1(s,z,y)^2+\cdots+\psi^{d-1}(s,z,y)^2+\|\nabla f\|^2\|\psi_Z(s,z,y)\|^2\right)\right) \nu_Z(y)\,dy\,ds \\ \le\ &\int_0^t \int_{\|y\|>1} \left(1\wedge (2\vee\|\nabla f\|^2)\|\psi_Z(s,z,y)\|^2\right) \nu_Z(y)\,dy\,ds. \end{aligned}$$

Given the condition (2.39), one may apply Lemma 2.2 and express $\mathcal{K}_t$ as $\mathcal{K}_t=\int_0^t\int y\tilde{M}(ds\,dy)$ where $\tilde{M}$ is a compensated integer-valued random measure on $[0,T]\times\mathbb{R}^d$ with compensator $m(t,Z_{t-},y)\,dy\,dt$.
Extracting the d-th component of $\mathcal{K}_t$, one obtains the semimartingale decomposition of ξ_t on $[0,T]$

$$\xi_t=\xi_0+\int_0^t \beta_s\,ds+\int_0^t \delta_s\,dB_s+\int_0^t\int u\,\tilde{K}(ds\,du),$$

where

$$\begin{cases} \beta_t &= \nabla f(Z_{t^-}).b_Z(t,Z_{t-})+\frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_{t^-})^t a_Z(t,Z_{t-})a_Z(t,Z_{t-})\right] \\ &\quad +\int_{\mathbb{R}^d}\left(f(Z_{t^-}+\psi_Z(t,Z_{t-},y))-f(Z_{t^-})-\psi_Z(t,Z_{t-},y).\nabla f(Z_{t^-})\right)\nu_Z(y)\,dy, \\ \delta_t &= \|\nabla f(Z_{t-})a_Z(t,Z_{t-})\|, \end{cases}$$

and K is an integer-valued random measure on $[0,T] \times \mathbb{R}$ with compensator $k(t, Z_{t-}, u)\, du\, dt$ defined for all $z \in \mathbb{R}^d$ via

$$\begin{aligned} k(t,z,u) &= \int_{\mathbb{R}^{d-1}} m(t,(y_1,\cdots,y_{d-1},u),z)\, dy_1 \cdots dy_{d-1} \\ &= \int_{\mathbb{R}^{d-1}} |\mathrm{det}\nabla_y \Phi(t,z,(y_1,\cdots,y_{d-1},u))|\ \nu_Z(\Phi(t,z,(y_1,\cdots,y_{d-1},u)))\, dy_1 \cdots dy_{d-1}, \end{aligned}$$

and $\tilde{K}$ its compensated random measure. In particular for any $u > 0$ (and analogously for $u < 0$),

$$k(t,z,[u,\infty[) = \int_{\mathbb{R}^d} 1_{\{f(z+\psi_Z(t,z,y))-f(z)\geq u\}}\, \nu_Z(y)\, dy.$$

$\square$

Given the semimartingale decomposition of ξ_t in the form (2.2), we may now construct the Markovian projection of ξ as follows.

Theorem 2.5. *Assume that*

- *the coefficients (b_Z, a_Z, m_Z) satisfy Assumptions 2.1, 2.2 and 2.3,*
- *the Markov process Z has a transition density $q_t(.)$ which is continuous on $\mathbb{R}^d$ uniformly in $t \in [0,T]$, and $t \mapsto q_t(z)$ is right-continuous on $[0,T[$, uniformly in $z \in \mathbb{R}^d$.*
- *$f \in C_b^2(\mathbb{R}^d, \mathbb{R})$ such that*

$$\forall (z_1,\cdots,z_{d-1}) \in \mathbb{R}^{d-1}, \quad u \mapsto f(z_1,\ldots,z_{d-1},u) \text{ is a } \mathcal{C}^1(\mathbb{R},\mathbb{R})-\text{diffeomorphism.}$$

Define, for $w \in \mathbb{R}, t \in [0,T]$,

$$
\begin{aligned}
b(t,w) = \frac{1}{c(w)} \int_{\mathbb{R}^{d-1}} & \Big[\nabla f(.).b_Z(t,.) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(.)^t a_Z(t,.) a_Z(t,.)\right] \\
& + \int_{\mathbb{R}^d} \left(f(. + \psi_Z(t,.,y)) - f(.) - \psi_Z(t,.,y).\nabla f(.)\right) \nu_Z(y)\, dy, \Big](z_1,\cdots,z_{d-1},w) \\
& \qquad \times \frac{q_t(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}\right|(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}, \\
\sigma(t,w) = \frac{1}{\sqrt{c(w)}} \Big[& \int_{\mathbb{R}^{d-1}} \|\nabla f(.) a_Z(t,.)\|^2 (z_1,\cdots,z_{d-1},w) \\
& \qquad \times \frac{q_t(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}\right|(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))} \Big]^{1/2}, \\
j(t,[u,\infty[,w) = \frac{1}{c(w)} \int_{\mathbb{R}^{d-1}} & \left(\int_{\mathbb{R}^d} 1_{\{f(.+\psi_Z(t,.,y)) - f(.) \geq u\}}(z_1,\cdots,z_{d-1},w)\, \nu_Z(y)\, dy \right) \\
& \qquad \times \frac{q_t(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))\right|},
\end{aligned}
\tag{2.47}
$$

for $u > 0$ (and analogously for $u < 0$), with

$$
c(w) = \int_{\mathbb{R}^{d-1}} dz_1 ... dz_{d-1} \frac{q_t(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))\right|}.
$$

Then the stochastic differential equation

$$
\begin{aligned}
X_t = \xi_0 &+ \int_0^t b(s,X_s)\, ds + \int_0^t \sigma(s,X_s)\, dB_s \\
&+ \int_0^t \int_{\|y\| \leq 1} y\, \tilde{J}(ds\, dy) + \int_0^t \int_{\|y\|>1} y\, J(ds\, dy),
\end{aligned}
\tag{2.48}
$$

where (B_t) is a Brownian motion, J is an integer-valued random measure on $[0,T] \times \mathbb{R}$ with compensator $j(t,du,X_{t-})\,dt$, $\tilde{J} = J - j$, admits a weak solution $((X_t)_{t\in[0,T]}, \mathbb{Q}_{\xi_0})$, unique in law, whose marginal distributions mimick those of ξ:

$$
\forall t \in [0,T], \quad X_t \overset{\mathrm{d}}{=} \xi_t.
$$

Under $\mathbb{Q}_{\xi_0}$, X is a Markov process with infinitesimal generator $\mathcal{L}$ given by

$$\forall g \in \mathcal{C}_0^\infty(\mathbb{R}) \quad \mathcal{L}_t g(w) = b(t,w)g'(w) + \frac{\sigma^2(t,w)}{2}\, g''(w) + \int_{\mathbb{R}^d} [g(w+u) - g(w) - ug'(w)]j(t,du,w).$$

Before proving Theorem 2.5, we start with an useful Lemma, which will be of importance.

Lemma 2.3. *Let Z be a $\mathbb{R}^d$-valued random variable with density $q(z)$ and $f \in \mathcal{C}^1(\mathbb{R}^d,\mathbb{R})$ such that*

$$\forall (z_1,\cdots,z_{d-1}) \in \mathbb{R}^{d-1}, \quad u \mapsto f(z_1,\ldots,z_{d-1},u) \text{ is a } \mathcal{C}^1(\mathbb{R},\mathbb{R})-\text{diffeomorphism.}$$

Define the function $F : \mathbb{R}^d \to \mathbb{R}$ such that $f(z_1,\cdots,z_{d-1},F(z)) = z_d$. Then for any measurable function $g : \mathbb{R}^d \to \mathbb{R}$ such that $\mathbb{E}\left[|g(Z)|\right] < \infty$ and any $w \in \mathbb{R}$,

$$\begin{aligned}
&\mathbb{E}\left[g(Z)|f(Z)=w\right] \\
= \ &\frac{1}{c(w)} \int_{\mathbb{R}^{d-1}} dz_1...dz_{d-1} g(z_1,\cdots,z_{d-1},w) \frac{q(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))\right|},
\end{aligned}$$

with

$$c(w) = \int_{\mathbb{R}^{d-1}} dz_1...dz_{d-1} \frac{q(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{\left|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))\right|}.$$

Proof. Consider the d-dimensional random variable $\kappa(Z)$, where $\kappa : \mathbb{R}^d \mapsto \mathbb{R}^d$ is given by

$$\kappa(z) = (z_1,\cdots,z_{d-1},f(z)).$$

$$(\nabla_z \kappa) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \frac{\partial f}{\partial z_1} & \cdots & \frac{\partial f}{\partial z_{d-1}} & \frac{\partial f}{\partial z_d} \end{pmatrix},$$

so t $|\det(\nabla_z \kappa)|(z) = \left|\frac{\partial f}{\partial z_d}(z)\right| > 0$. Hence κ is a $\mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d)$-diffeomorphism with inverse κ^{-1}.

$$\kappa(\kappa^{-1}(z)) = (\kappa_1^{-1}(z),\cdots,\kappa_{d-1}^{-1}(z), f(\kappa_1^{-1}(z),\cdots,\kappa_d^{-1}(z)) = z.$$

For $1 \leq i \leq d-1$, $\kappa_i^{-1}(z) = z_i$ and $f(z_1, \cdots, z_{d-1}, \kappa_d^{-1}(z)) = z_d$ that is $\kappa_d^{-1}(z) = F(z)$. Hence

$$\kappa^{-1}(z_1, \cdots, z_d) = (z_1, \cdots, z_{d-1}, F(z)).$$

Define $q_\kappa(z)\, dz$ the inverse image of the measure $q(z)\, dz$ under the partial map κ by

$$\begin{aligned} q_\kappa(z) &= 1_{\{\kappa(\mathbb{R}^d)\}}(z)\, |\det(\nabla_z \kappa^{-1})|(z)\, q(\kappa^{-1}(z)) \\ &= 1_{\{\kappa(\mathbb{R}^d)\}}(z) \left| \frac{\partial f}{\partial z_d} \right|^{-1} (z_1, \cdots, z_{d-1}, F(z))\, q(z_1, \cdots, z_{d-1}, F(z)). \end{aligned}$$

$q_\kappa(z)$ is the density of $\kappa(Z)$. So, for any $w \in f(\mathbb{R}^d) = \mathbb{R}$,

$$\begin{aligned} & \mathbb{E}\left[g(Z) | f(Z) = w\right] \\ = & \int_{\mathbb{R}^{d-1}} \mathbb{E}\left[g(Z) | \kappa(Z) = (z_1, \cdots, z_{d-1}, w)\right]\, dz_1, \cdots, dz_{d-1} \\ = & \frac{1}{c(w)} \int_{\mathbb{R}^{d-1}} dz_1 ... dz_{d-1} g(z_1, \cdots, z_{d-1}, w) \frac{q(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}{\left|\frac{\partial f}{\partial z_d}\right| (z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}, \end{aligned}$$

with

$$c(w) = \int_{\mathbb{R}^{d-1}} dz_1 ... dz_{d-1} \frac{q(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))}{\left|\frac{\partial f}{\partial z_d}(z_1, \cdots, z_{d-1}, F(z_1, \cdots, z_{d-1}, w))\right|}.$$

□

Proof of Theorem 2.5. Let us show that if (b_Z, a_Z, m_Z) satisfy Assumptions 2.1, 2.2 and 2.3 then the triplet $(\delta_t, \beta_t, k(t, Z_{t-}, u))$ satisfies the assumptions of Theorem 2.2. Then given Proposition 2.3, one may build in an explicit manner the Markovian Projection of ξ_t.

First, note that β_t and δ_t satisfy Assumption 2.4 since $b_Z(t,z)$ and $a_Z(t,z)$ satisfy Assumption 2.1 and ∇f and $\nabla^2 f$ are bounded.

One observes that if m_Z satisfies Assumption 2.1, then the equality (2.41) implies that ψ_Z and ν_Z satisfies:

$$\exists K_2 > 0\, \forall t \in [0,T] \forall z \in \mathbb{R}^d \int_0^t \int_{\{\|y\| \geq 1\}} \left(1 \wedge \|\psi_Z(s,z,y)\|^2\right)\, \nu_Z(y)\, dy\, ds < K_2. \tag{2.49}$$

Hence,

$$\begin{aligned}\int_0^t \int \left(1 \wedge \|u\|^2\right) k(s, Z_{s-}, u)\, du\, ds &= \int_0^t \int \left(1 \wedge |f(Z_{s^-} + \psi_Z(s, Z_{s-}, y)) - f(Z_{s^-})|^2\right) \nu_Z(y)\, dy\, ds \\ &\leq \int_0^t \int \left(1 \wedge \|\nabla f\|^2 \|\psi_Z(s, Z_{s-}, u)\|^2\right) \nu_Z(y)\, dy\, ds,\end{aligned}$$

is bounded and k satisfies Assumption 2.5.

As argued before, one sees that if a_Z is non-degenerate then δ_t is. In the case $\delta_t \equiv 0$, for $t \in [0, T[, R > 0, z \in B(0, R)$ and $g \in \mathcal{C}_0^0(\mathbb{R}) \geq 0$, denoting C and $K_T > 0$ the constants in Assumption 2.3,

$$\begin{aligned}&k(t, z, u) \\ = \;& \int_{\mathbb{R}^{d-1}} |\det \nabla_y \Phi(t, z, (y_1, \cdots, y_{d-1}, u))|\; \nu_Z(\Phi(t, z, (y_1, \cdots, y_{d-1}, u)))\, dy_1 \cdots dy_{d-1} \\ = \;& \int_{\mathbb{R}^{d-1}} \left|\det \nabla_y \phi(t, z, \kappa_z^{-1}(y_1, \cdots, y_{d-1}, u))\right| \left|\frac{\partial f}{\partial y_d}(z + \kappa_z^{-1}(y_1, \cdots, y_{d-1}, u))\right|^{-1} \\ &\qquad\qquad \nu_Z(\phi(t, z, \kappa_z^{-1}(y_1, \cdots, y_d, u)))\, dy_1 \cdots dy_{d-1} \\ \geq \;& \int_{\mathbb{R}^{d-1}} \left|\frac{\partial f}{\partial y_d}(z + \kappa_z^{-1}(y_1, \cdots, y_{d-1}, u))\right|^{-1} \frac{C}{\|\kappa_z^{-1}(y_1, \cdots, y_{d-1}, u)\|^{d+\beta}}\, dy_1 \cdots dy_{d-1} \\ = \;& \int_{\mathbb{R}^{d-1}} \frac{C}{\|(y_1, \cdots, y_{d-1}, u)\|^{d+\beta}}\, dy_1 \cdots dy_{d-1} \\ = \;& \frac{1}{|u|^{d+\beta}} \int_{\mathbb{R}^{d-1}} \frac{C}{\|(y_1/u, \cdots, y_{d-1}/u, 1)\|^{d+\beta}}\, dy_1 \cdots dy_{d-1} = C' \frac{1}{|u|^{1+\beta}},\end{aligned}$$

with $C' = \int_{\mathbb{R}^{d-1}} C\, \|(w_1, \cdots, w_{d-1}, 1)\|^{-1}\, dw_1 \cdots dw_{d-1}$.

Similarly

$$
\begin{aligned}
&\int \left(1 \wedge |u|^{\beta}\right) \left(k(t,z,u) - \frac{C'}{|u|^{1+\beta}}\right) du \\
= \;& \int \left(1 \wedge |u|^{\beta}\right) \int_{\mathbb{R}^{d-1}} \left|\frac{\partial f}{\partial y_d}(z + \kappa_z^{-1}(y_1, \cdots, y_{d-1}, u))\right|^{-1} \\
& \qquad \Big[\left|\det \nabla_y \phi(t, z, \kappa_z^{-1}(y_1, \cdots, y_{d-1}, u))\right| \nu_Z(\phi(t, z, \kappa_z^{-1}(y_1, \cdots, y_{d-1}, u))) \\
& \qquad\qquad - \frac{C}{\|\kappa_z^{-1}(y_1, \cdots, y_{d-1}, u)\|^{d+\beta}} \Big] \, dy_1 \cdots dy_{d-1} \, du \\
= \;& \int_{\mathbb{R}^d} \left(1 \wedge |f(z + (y_1, \cdots, y_{d-1}, u)) - f(z)|^{\beta}\right) \\
& \qquad \Big(\left|\det \nabla_y \phi(t, z, (y_1, \cdots, y_{d-1}, u))\right| \nu_Z(\phi(t, z, (y_1, \cdots, y_{d-1}, u))) \\
& \qquad\qquad - \frac{C}{\|(y_1, \cdots, y_{d-1}, u)\|^{d+\beta}} \Big) \, dy_1 \cdots dy_{d-1} \, du \\
\leq \;& \int_{\mathbb{R}^d} \left(1 \wedge \|\nabla f\| \|(y_1, \cdots, y_{d-1}, u)\|^{\beta}\right) \\
& \Big(\left|\det \nabla_y \phi(t, z, (y_1, \cdots, y_{d-1}, u))\right| \nu_Z(\phi(t, z, (y_1, \cdots, y_{d-1}, u))) \\
& - \frac{C}{\|(y_1, \cdots, y_{d-1}, u)\|^{d+\beta}} \Big) \, dy_1 \cdots dy_{d-1} \, du
\end{aligned}
$$

is also bounded. Similar arguments would show that

$$
\begin{aligned}
& \lim_{\epsilon \to 0} \int_{|u| \leq \epsilon} |u|^{\beta} \left(k(t, Z_{t-}, u) - \frac{C}{|u|^{1+\beta}}\right) du = 0 \text{ a.s.} \\
\text{and} \quad & \lim_{R \to \infty} \int_0^T k\left(t, Z_{t-}, \{|u| \geq R\}\right) \, dt = 0 \text{ a.s.},
\end{aligned}
$$

since this essentially hinges on the fact that f has bounded derivatives.

Applying Lemma 2.3, one can compute explicitly the conditional expectations in (2.36). For example,

$$\begin{aligned} b(t,w) &= E[\beta_t|\xi_{t-}=w] = \int_{\mathbb{R}^{d-1}} \Big[\nabla f(.).b_Z(t,.) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(.)^t a_Z(t,.) a_Z(t,.)\right] \\ &\quad + \int_{\mathbb{R}^d} \left(f(.+\psi_Z(t,.,y)) - f(.) - \psi_Z(t,.,y).\nabla f(.)\right) \nu_Z(y)\,dy, \Big](z_1,\cdots,z_{d-1},w) \\ &\quad \times \frac{q_t(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))}{|\frac{\partial f}{\partial z_d}(z_1,\cdots,z_{d-1},F(z_1,\cdots,z_{d-1},w))|}. \end{aligned}$$

with $F:\mathbb{R}^d \to \mathbb{R}$ defined by $f(z_1,\cdots,z_{d-1},F(z)) = z_d$. Furthermore, f is C^2 with bounded derivatives, (b_Z, a_Z, ν_Z) satisfy Assumption 2.1. Since $z \in \mathbb{R}^d \to q_t(z)$ is continuous in z uniformly in $t \in [0,T]$ and $t \in [0,T[\to q_t(z)$ is right-continuous in t uniformly in $z \in \mathbb{R}^d$, the same properties hold for b. Proceeding similarly, one can show that that Assumption 2.2 holds for σ and j so Theorem 2.2 may be applied to yield the result. □

2.4.2 Time changed Lévy processes

Models based on time–changed Lévy processes have been the focus of much recent work, especially in mathematical finance [24]. Let L_t be a Lévy process, (b,σ^2,ν) be its characteristic triplet on $(\Omega,(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$, N the Poisson random measure representing the jumps of L and $(\theta_t)_{t\geq 0}$ a locally bounded, strictly positive $\mathcal{F}_t$-adapted cadlag process. The process

$$\xi_t = \xi_0 + L_{\Theta_t} \qquad \Theta_t = \int_0^t \theta_s ds.$$

is called a time-changed Lévy process where θ_t is interpreted as the rate of time change.

Theorem 2.6 (Markovian projection of time-changed Lévy processes)**.** *Assume that $(\theta_t)_{t\geq 0}$ is bounded from above and away from zero:*

$$\exists K,\epsilon > 0, \forall t \in [0,T],\ \ K \geq \theta_t \geq \epsilon \qquad \text{a.s.} \tag{2.50}$$

and that there exists $\alpha : [0,T]\times\mathbb{R} \mapsto \mathbb{R}$ such that

$$\forall t \in [0,T], \quad \forall z \in \mathbb{R}, \qquad \alpha(t,z) = E[\theta_t|\xi_{t-}=z],$$

where $\alpha(t,.)$ is continuous on $\mathbb{R}$, uniformly in $t \in [0,T]$ and, for all z in $\mathbb{R}$, $\alpha(.,z)$ is right-continuous in t on $[0,T[$.

$$\begin{array}{lll} \text{If either} & (i) & \sigma > 0 \\ \text{or} & (ii) & \sigma \equiv 0 \quad \text{and}\ \exists \beta \in]0,2[, c, K' > 0, \text{ and a measure} \end{array}$$

$$\nu^\beta(dy) \text{ such that } \quad \nu(dy) = \nu^\beta(dy) + \frac{c}{|y|^{1+\beta}}\, dy,$$

$$\int \left(1 \wedge |y|^\beta\right) \nu^\beta(dy) \leq K', \quad \lim_{\epsilon \to 0} \int_{|y| \leq \epsilon} |y|^\beta \, \nu^\beta(dy) = 0,$$

then

- *(ξ_t) has the same marginals as (X_t) on $[0,T]$, defined as the weak solution of*

$$X_t = \xi_0 + \int_0^t \sigma \sqrt{\alpha(s, X_{s-})} dB_s + \int_0^t b\alpha(s, X_{s-}) ds + \int_0^t \int_{|z|\leq 1} z \tilde{J}(ds\, dz) + \int_0^t \int_{|z|>1} z J(ds\, dz),$$

  where B_t is a real-valued brownian motion, J is an integer-valued random measure on $[0,T] \times \mathbb{R}$ with compensator $\alpha(t, X_{t-})\, \nu(dy)\, dt$.

- *The marginal distribution p_t of ξ_t is the unique solution of the forward equation:*

$$\frac{\partial p_t}{\partial t} = \mathcal{L}_t^\star . \, p_t,$$

  where, $\mathcal{L}_t^*$ is given by

$$\begin{aligned} \mathcal{L}_t^\star g(x) &= -b \frac{\partial}{\partial x} [\alpha(t,x) g(x)] + \frac{\sigma^2}{2} \frac{\partial^2}{\partial x^2} [\alpha^2(t,x) g(x)] \\ &+ \int_{\mathbb{R}^d} \nu(dz) \left[g(x-z)\alpha(t,x-z) - g(x)\alpha(t,x) - 1_{\|z\|\leq 1} z . \frac{\partial}{\partial x}[g(x)\alpha(t,x)] \right], \end{aligned}$$

  with the given initial condition $p_0(dy) = \mu_0(dy)$ where μ_0 denotes the law of ξ_0.

Proof. Consider the Lévy-Ito decomposition of L:

$$L_t = bt + \sigma W_t + \int_0^t \int_{|z|\leq 1} z\tilde{N}(dsdz) + \int_0^t \int_{|z|>1} zN(dsdz).$$

Then ξ rewrites

$$\begin{aligned}\xi_t &= \xi_0 + \sigma W(\Theta_t) + b\Theta_t \\ &+ \int_0^{\Theta_t} \int_{|z|\leq 1} z\tilde{N}(ds\,dz) + \int_0^{\theta_t} \int_{|z|>1} zN(ds\,dz).\end{aligned}$$

$W(\Theta_t)$ is a continuous martingale starting from 0, with quadratic variation $\Theta_t = \int_0^t \theta_s ds$. Hence, there exists Z_t a Brownian motion, such that

$$W(\Theta_t) \stackrel{d}{=} \int_0^t \sqrt{\theta_s} dZ_s.$$

Hence ξ_t is the weak solution of :

$$\begin{aligned}\xi_t &= \xi_0 + \int_0^t \sigma\sqrt{\theta_s}\, dZ_s + \int_0^t b\theta_s\, ds \\ &+ \int_0^t \int_{|z|\leq 1} z\theta_s\, \tilde{N}(ds\,dz) + \int_0^t \int_{|z|>1} z\theta_s\, N(ds\,dz).\end{aligned}$$

Using the notations of Theorem 2.2,

$$\beta_t = b\,\theta_t, \quad \delta_t = \sigma\,\sqrt{\theta_t}, \quad m(t,dy) = \theta_t\,\nu(dy).$$

Given the conditions (2.50), one simply observes that

$$\forall (t,z) \in [0,T]\times\mathbb{R}, \quad \epsilon \leq \alpha(t,z) \leq K.$$

Hence Assumptions 2.4, 2.5 and 2.6 hold for (β,δ,m). Furthermore,

$$\begin{aligned} b(t,.) &= \mathbb{E}\left[\beta_t|\xi_{t^-} = .\right] = b\,\alpha(t,.), \\ \sigma(t,.) &= \mathbb{E}\left[\delta_t^2|\xi_{t^-} = .\right]^{1/2} = \sigma\,\sqrt{\alpha(t,.)}, \\ n(t,B,.) &= \mathbb{E}\left[m(t,B)|\xi_{t^-} = .\right] = \alpha(t,.)\nu(B), \end{aligned}$$

are all continuous on $\mathbb{R}$ uniformly in t on $[0,T]$ and for all $z\in\mathbb{R}$, $\alpha(.,z)$ is right-continuous on $[0,T[$. One may apply Theorem 2.2 yielding the result. □

The impact of the random time change on the marginals can be captured by making the characteristics state dependent

$$(\ b\alpha(t, X_{t-}), \sigma^2\alpha(t, X_{t-}), \alpha(t, X_{t-})\nu \ )$$

by introducing the *same* adjustment factor $\alpha(t, X_{t-})$ to the drift, diffusion coefficient and Lévy measure. In particular if $\alpha(t, x)$ is affine in x we get an affine process [32] where the affine dependence of the characteristics with respect to the state are restricted to be colinear, which is rather restrictive. This remark shows that time-changed Lévy processes, which in principle allow for a wide variety of choices for θ and L, may not be as flexible as apparently simpler affine models when it comes to reproducing marginal distributions.

Chapter 3

Forward equations for option prices in semimartingale models

Since the seminal work of Black, Scholes and Merton [20, 75] partial differential equations (PDE) have been used as a way of characterizing and efficiently computing option prices. In the Black-Scholes-Merton model and various extensions of this model which retain the Markov property of the risk factors, option prices can be characterized in terms of solutions to a backward PDE, whose variables are time (to maturity) and the value of the underlying asset. The use of backward PDEs for option pricing has been extended to cover options with path-dependent and early exercise features, as well as to multifactor models (see e.g. [1]). When the underlying asset exhibit jumps, option prices can be computed by solving an analogous partial integro-differential equation (PIDE) [4, 31].

A second important step was taken by Dupire [33, 34, 36] who showed that when the underlying asset is assumed to follow a diffusion process

$$dS_t = S_t\sigma(t, S_t)dW_t$$

prices of call options (at a given date t_0) solve a *forward* PDE

$$\frac{\partial C_{t_0}}{\partial T}(T,K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\,\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K)$$

on $[t_0,\infty[\times]0,\infty[$ in the strike and maturity variables, with the initial condition

$$\forall K > 0 \quad C_{t_0}(t_0,K) = (S_{t_0} - K)_+.$$

This forward equation allows to price call options with various strikes and maturities on the same underlying asset, by solving a *single* partial differential equation. Dupire's forward equation also provides useful insights into the *inverse problem* of calibrating diffusion models to observed call and put option prices [16].

Given the theoretical and computational usefulness of the forward equation, there have been various attempts to extend Dupire's forward equation to other types of options and processes, most notably to Markov processes with jumps [4, 26, 29, 62, 25]. Most of these constructions use the Markov property of the underlying process in a crucial way (see however [65]).

As noted by Dupire [35], the forward PDE holds in a more general context than the backward PDE: even if the (risk-neutral) dynamics of the underlying asset is not necessarily Markovian, but described by a continuous Brownian martingale

$$dS_t = S_t \delta_t dW_t,$$

then call options still verify a forward PDE where the diffusion coefficient is given by the local (or effective) volatility function $\sigma(t, S)$ given by

$$\sigma(t, S) = \sqrt{E[\delta_t^2 | S_t = S]}.$$

This method is linked to the "Markovian projection" problem: the construction of a Markov process which mimicks the marginal distributions of a martingale (see Chapter 2 and [51, 74]). Such "mimicking processes" provide a method to extend the Dupire equation to non-Markovian settings.

We show in this work that the forward equation for call prices holds in a more general setting, where the dynamics of the underlying asset is described by a – possibly discontinuous – semimartingale. Our parametrization of the price dynamics is general, allows for stochastic volatility and does *not* assume jumps to be independent or driven by a Lévy process, although it includes these cases. Also, our derivation does not require ellipticity or nondegeneracy of the diffusion coefficient. The result is thus applicable to various stochastic volatility models with jumps, pure jump models and point process models used in equity and credit risk modeling.

Our result extends the forward equation from the original diffusion setting of Dupire [34] to various examples of non-Markovian and/or discontinuous processes and implies previous derivations of forward equations [4, 26, 25, 29, 34, 35, 62, 73] as special cases. Section 3.2 gives examples of forward

PIDEs obtained in various settings: time-changed Lévy processes, local Lévy models and point processes used in portfolio default risk modeling. In the case where the underlying risk factor follows, an Itô process or a Markovian jump-diffusion driven by a Lévy process, we retrieve previously known forms of the forward equation. In this case, our approach gives a rigorous derivation of these results under precise assumptions in a unified framework. In some cases, such as index options (Sec. 3.2.5) or CDO expected tranche notionals (Sec. 3.2.6), our method leads to a new, more general form of the forward equation valid for a larger class of models than previously studied [5, 29, 85].

The forward equation for call options is a PIDE in one (spatial) dimension, regardless of the number of factors driving the underlying asset. It may thus be used as a method for reducing the dimension of the problem. The case of index options (Section 3.2.5) in a multivariate jump-diffusion model illustrates how the forward equation projects a high dimensional pricing problem into a one-dimensional state equation.

3.1 Forward PIDEs for call options

3.1.1 General formulation of the forward equation

Consider a (strictly positive) semimartingale S whose dynamics under the pricing measure $\mathbb{P}$ is given by

$$S_T = S_0 + \int_0^T r(t) S_{t^-} dt + \int_0^T S_{t^-} \delta_t dW_t + \int_0^T \int_{-\infty}^{+\infty} S_{t^-} (e^y - 1) \tilde{M}(dt\ dy), \tag{3.1}$$

where $r(t) > 0$ represents a (deterministic) bounded discount rate, δ_t the (random) volatility process and M is an integer-valued random measure with compensator

$$\mu(dt\, dy; \omega) = m(t, dy, \omega)\, dt,$$

representing jumps in the log-price, and $\tilde{M} = M - \mu$ is the compensated random measure associated to M (see [30] for further background). Both the volatility δ_t and $m(t, dy)$, which represents the intensity of jumps of size y at time t, are allowed to be stochastic. In particular, we do *not* assume the jumps to be driven by a Lévy process or a process with independent increments. The specification (3.1) thus includes most stochastic volatility models with jumps.

We assume the following conditions:

Assumption 3.1 (Full support)**.** $\forall t \geq 0$, $\mathrm{supp}(S_t) = [0, \infty[$.

Assumption 3.2 (Integrability condition)**.**

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2\, dt + \int_0^T dt \int_{\mathbb{R}} (e^y - 1)^2 m(t, dy)\right)\right] < \infty. \qquad \text{(H)}$$

The value $C_{t_0}(T, K)$ at t_0 of a call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T, K) = e^{-\int_{t_0}^T r(t)\, dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}]. \qquad (3.2)$$

Under Assumption (H), Remark 3.2 (see below) implies that the expectation in (3.2) is finite. Our main result is the following:

Theorem 3.1 (Forward PIDE for call options)**.** *Let ψ_t be the exponential double tail of the compensator $m(t, dy)$*

$$\psi_t(z) = \begin{cases} \int_{-\infty}^z dx\ e^x \int_{-\infty}^x m(t, du), & z < 0\ ; \\ \int_z^{+\infty} dx\ e^x \int_x^{\infty} m(t, du), & z > 0 \end{cases} \qquad (3.3)$$

and let $\sigma : [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$, $\chi : [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$ be measurable functions such that for all $t \in [t_0, T]$

$$\begin{cases} \sigma(t, S_{t-}) &= \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t-}\right]}; \\ \chi_{t, S_{t-}}(z) &= \mathbb{E}\left[\psi_t\left(z\right) | S_{t-}\right] \qquad a.s. \end{cases} \qquad (3.4)$$

Under assumption (H)*, the call option price $(T, K) \mapsto C_{t_0}(T, K)$, as a function of maturity and strike, is a weak solution (in the sense of distributions) of the partial integro-differential equation*

$$\begin{aligned} \frac{\partial C_{t_0}}{\partial T}(T, K) = -r(T)K\frac{\partial C_{t_0}}{\partial K}(T, K) + \frac{K^2 \sigma(T, K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T, K) \\ + \int_0^{+\infty} y \frac{\partial^2 C_{t_0}}{\partial K^2}(T, dy)\, \chi_{T, y}\left(\ln\left(\frac{K}{y}\right)\right) \end{aligned} \qquad (3.5)$$

on $]t_0, \infty[\times]0, \infty[$ and verifies the initial condition:

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

Remark 3.1 (Weak solutions). *Let $C_0^\infty(]t_0,\infty[\times]0,\infty[,\mathbb{R})$ be the set of infinitely differentiable functions with compact support in $]t_0,\infty[\times]0,\infty[$. Recall that a function $f:[t_0,\infty[\times]0,\infty[\mapsto\mathbb{R}$ is a weak solution of* (3.5) *on $]t_0,\infty[\times]0,\infty[$ in the sense of distributions if for any test function $\varphi\in C_0^\infty(]t_0,\infty[\times]0,\infty[,\mathbb{R})$,*

$$
\begin{aligned}
&-\int_{t_0}^{\infty} dt \int_0^\infty dK f(t,K)\,\frac{\partial\varphi}{\partial t}(t,K) = \\
&\int_{t_0}^{\infty} dt \int_0^\infty dK \varphi(t,K)\left[r(t)K\frac{\partial f}{\partial K} + \frac{K^2\sigma(t,K)^2}{2}\,\frac{\partial^2 f}{\partial K^2} + \int_0^{+\infty} y\frac{\partial^2 f}{\partial K^2}(t,dy)\,\chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right)\right].
\end{aligned}
$$

In our case, $K\mapsto C_{t_0}(t,K)$ is in fact a continuous and convex function for each $t\geq 0$. When $K\mapsto f(t,K)$ is a convex function for each $t\geq 0$, $\partial f/\partial K$ is defined (Lebesgue) almost-everywhere and $\partial^2 f/\partial K^2$ is in fact a measure so the right hand side is well defined without any differentiability requirement on the coefficients of the operator.

This notion of weak solution allows to separate the discussion of existence of solutions from the discussion of their regularity (which may be delicate, see [31]).

Remark 3.2. *The discounted asset price*

$$\hat{S}_T = e^{-\int_0^T r(t)dt}\,S_T,$$

is the stochastic exponential of the martingale U defined by

$$U_T = \int_0^T \delta_t\,dW_t + \int_0^T \int (e^y-1)\tilde{M}(dt\,dy).$$

Under assumption (H)*, we have*

$$\forall T>0,\quad \mathbb{E}\left[\exp\left(\frac{1}{2}\langle U,U\rangle_T^d + \langle U,U\rangle_T^c\right)\right] < \infty,$$

where $\langle U,U\rangle^c$ and $\langle U,U\rangle^d$ denote the continuous and discontinuous parts of the (increasing) process $[U]$. [80, Theorem 9] implies that $(\hat{S}_T)$ is a $\mathbb{P}$-martingale.

The form of the integral term in (3.5) may seem different from the integral term appearing in backward PIDEs [31, 54]. The following lemma expresses $\chi_{T,y}(z)$ in a more familiar form in terms of call payoffs:

Lemma 3.1. *Let $n(t,dz,y)\,dt$ be a measure on $[0,T]\times\mathbb{R}\times\mathbb{R}^+$ verifying*

$$\forall t\in[0,T],\qquad \int_{-\infty}^{\infty}(e^z\wedge|z|^2)n(t,dz,y)<\infty \qquad \text{a.s.}$$

Then the exponential double tail $\chi_{t,y}(z)$ of n, defined as

$$\chi_{t,y}(z)=\begin{cases}\int_{-\infty}^{z}dx\ e^x\int_{-\infty}^{x}n(t,du,y), & z<0\ ;\\ \int_{z}^{+\infty}dx\ e^x\int_{x}^{\infty}n(t,du,y), & z>0\end{cases} \tag{3.6}$$

verifies

$$\int_{\mathbb{R}}[(ye^z-K)^+-e^z(y-K)^+-K(e^z-1)1_{\{y>K\}}]n(t,dz,y)=y\,\chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right).$$

Proof. Let $K,T>0$. Then

$$\begin{aligned}&\int_{\mathbb{R}}[(ye^z-K)^+-e^z(y-K)^+-K(e^z-1)1_{\{y>K\}}]n(t,dz,y)\\ =\ &\int_{\mathbb{R}}[(ye^z-K)1_{\{z>\ln(\frac{K}{y})\}}-e^z(y-K)1_{\{y>K\}}-K(e^z-1)1_{\{y>K\}}]n(t,dz,y)\\ =\ &\int_{\mathbb{R}}[(ye^z-K)1_{\{z>\ln(\frac{K}{y})\}}+(K-ye^z)1_{\{y>K\}}]n(t,dz,y).\end{aligned}$$

- If $K\geq y$, then

$$\begin{aligned}&\int_{\mathbb{R}}1_{\{K\geq y\}}[(ye^z-K)1_{\{z>\ln(\frac{K}{y})\}}+(K-ye^z)1_{\{y>K\}}]n(t,dz,y)\\ =\ &\int_{\ln(\frac{K}{y})}^{+\infty}y(e^z-e^{\ln(\frac{K}{y})})\,n(t,dz,y).\end{aligned}$$

- If $K<y$, then

$$\begin{aligned}&\int_{\mathbb{R}}1_{\{K<y\}}[(ye^z-K)1_{\{z>\ln(\frac{K}{y})\}}+(K-ye^z)1_{\{y>K\}}]n(t,dz,y)\\ =\ &\int_{\ln(\frac{K}{y})}^{+\infty}[(ye^z-K)+(K-ye^z)]n(t,dz,y)+\int_{-\infty}^{\ln(\frac{K}{y})}[K-ye^z]n(t,dz,y)\\ =\ &\int_{-\infty}^{\ln(\frac{K}{y})}y(e^{\ln(\frac{K}{y})}-e^z)n(t,dz,y).\end{aligned}$$

Using integration by parts, $\chi_{t,y}$ can be equivalently expressed as

$$\chi_{t,y}(z) = \begin{cases} \int_{-\infty}^{z} (e^z - e^u)\, n(t, du, y), & z < 0\ ; \\ \int_{z}^{\infty} (e^u - e^z)\, n(t, du, y), & z > 0. \end{cases}$$

Hence

$$\int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y-K)^+ - K(e^z-1)1_{\{y>K\}}] n(t, dz, y) = y\, \chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right).$$

□

3.1.2 Derivation of the forward equation

In this section we present a proof of Theorem 3.1 using the Tanaka-Meyer formula for semimartingales [53, Theorem 9.43] under assumption (H).

Proof. We first note that, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_{t_0}}$ given $\mathcal{F}_{t_0}$, we may replace the conditional expectation in (3.2) by an expectation with respect to the marginal distribution $p_T^S(dy)$ of S_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we set $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets and we denote $C_0(T, K) \equiv C(T, K)$ for simplicity. (3.2) can be expressed as

$$C(T, K) = e^{-\int_0^T r(t)\, dt} \int_{\mathbb{R}^+} (y - K)^+\ p_T^S(dy). \tag{3.7}$$

By differentiating with respect to K, we obtain

$$\begin{aligned} \frac{\partial C}{\partial K}(T, K) &= -e^{-\int_0^T r(t)\, dt} \int_K^\infty p_T^S(dy) = -e^{-\int_0^T r(t)\, dt} \mathbb{E}\left[1_{\{S_T > K\}}\right], \\ \frac{\partial^2 C}{\partial K^2}(T, dy) &= e^{-\int_0^T r(t)\, dt} p_T^S(dy). \end{aligned} \tag{3.8}$$

Let $L_t^K = L_t^K(S)$ be the semimartingale local time of S at K under $\mathbb{P}$ (see [53, Chapter 9] or [81, Ch. IV] for definitions). Applying the Tanaka-Meyer formula to $(S_T - K)^+$, we have

$$\begin{aligned} (S_T - K)^+ = (S_0 - K)^+ &+ \int_0^T 1_{\{S_{t-} > K\}} dS_t + \frac{1}{2}(L_T^K) \\ &+ \sum_{0 < t \le T} \left[(S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-} > K\}} \Delta S_t\right]. \end{aligned} \tag{3.9}$$

As noted in Remark 3.2, the integrability condition (H) implies that the discounted price $\hat{S}_t = e^{-\int_0^t r(s)\,ds} S_t = \mathcal{E}(U)_t$ is a martingale under $\mathbb{P}$. So (3.1) can be expressed as

$$dS_t = e^{\int_0^t r(s)\,ds}\left(r(t)S_{t-}dt + d\hat{S}_t\right)$$

and

$$\int_0^T 1_{\{S_{t-}>K\}} dS_t = \int_0^T e^{\int_0^t r(s)\,ds}\, 1_{\{S_{t-}>K\}} d\hat{S}_t + \int_0^T e^{\int_0^t r(s)\,ds}\, r(t) S_{t-} 1_{\{S_{t-}>K\}} dt,$$

where the first term is a martingale. Taking expectations, we obtain

$$\begin{aligned} e^{\int_0^T r(t)\,dt} C(T,K) - (S_0-K)^+ &= \mathbb{E}\left[\int_0^T e^{\int_0^t r(s)\,ds}\, r(t) S_t\, 1_{\{S_{t-}>K\}} dt + \frac{1}{2} L_T^K\right] \\ &+ \mathbb{E}\left[\sum_{0<t\leq T}\left((S_t-K)^+ - (S_{t-}-K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t\right)\right]. \end{aligned}$$

Noting that $S_{t-} 1_{\{S_{t-}>K\}} = (S_{t-}-K)^+ + K 1_{\{S_{t-}>K\}}$, we obtain

$$\mathbb{E}\left[\int_0^T e^{\int_0^t r(s)\,ds}\, r(t) S_{t-} 1_{\{S_{t-}>K\}} dt\right] = \int_0^T r(t) e^{\int_0^t r(s)\,ds}\left[C(t,K) - K\frac{\partial C}{\partial K}(t,K)\right] dt,$$

using Fubini's theorem and (3.8). As for the jump term,

$$\begin{aligned} &\mathbb{E}\left[\sum_{0<t\leq T} (S_t-K)^+ - (S_{t-}-K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t\right] \\ = \; &\mathbb{E}\left[\int_0^T dt \int m(t,dx)\,(S_{t-}e^x-K)^+ - (S_{t-}-K)^+ - 1_{\{S_{t-}>K\}} S_{t-}(e^x-1)\right] \\ = \; &\mathbb{E}\Big[\int_0^T dt \int m(t,dx)\big((S_{t-}e^x-K)^+ - (S_{t-}-K)^+ \\ &\qquad -(S_{t-}-K)^+(e^x-1) - K 1_{\{S_{t-}>K\}}(e^x-1)\big)\Big]. \end{aligned}$$

Applying Lemma 3.1 to the random measure m we obtain that

$$\int m(t,dx)\big((S_{t-}e^x-K)^+ - e^x(S_{t-}-K)^+ - K 1_{\{S_{t-}>K\}}(e^x-1)\big) = S_{t-}\,\psi_t\left(\ln\left(\frac{K}{S_{t-}}\right)\right)$$

holds true. One observes that for all z in $\mathbb{R}$

$$\begin{aligned}
\psi_t(z) &\le 1_{\{z<0\}} \int_{-\infty}^{z} e^z\, m(t,du) + 1_{\{z>0\}} \int_{-\infty}^{z} e^u\, m(t,du) \\
&= 1_{\{z<0\}} e^z \int_{-\infty}^{z} 1.\, m(t,du) + 1_{\{z>0\}} \int_{-\infty}^{z} e^u\, m(t,du).
\end{aligned}$$

Using Assumption (H),

$$\begin{aligned}
\mathbb{E}\left[\sum_{0<t\le T} \left[(S_t-K)^+ - (S_{t-}-K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t\right]\right] &= \mathbb{E}\left[\int_0^T dt\, S_{t-}\, \psi_t\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \\
&< \infty.
\end{aligned}$$

Hence applying Fubini's theorem leads to

$$\begin{aligned}
&\mathbb{E}\left[\sum_{0<t\le T} (S_t-K)^+ - (S_{t-}-K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t\right] \\
&= \int_0^T dt \mathbb{E}\Big[\int m(t,dx)\big((S_{t-}e^x-K)^+ - e^x(S_{t-}-K)^+ - K 1_{\{S_{t-}>K\}}(e^x-1)\big)\Big] \\
&= \int_0^T dt\, \mathbb{E}\left[S_{t-}\, \psi_t\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right] \\
&= \int_0^T dt\, \mathbb{E}\left[S_{t-}\mathbb{E}\left[\psi_t\left(\ln\left(\frac{K}{S_{t-}}\right)\right)|S_{t-}\right]\right] \\
&= \int_0^T dt\, \mathbb{E}\left[S_{t-}\, \chi_{t,S_{t-}}\left(\ln\left(\frac{K}{S_{t-}}\right)\right)\right].
\end{aligned}$$

Let $\varphi \in C_0^\infty([0,T]\times]0,\infty[)$ be an infinitely differentiable function with compact support in $[0,T]\times]0,\infty[$. The extended occupation time formula [82, Chap. VI, Exercise 1.15] yields

$$\int_0^{+\infty} dK \int_0^T \varphi(t,K)\, dL_t^K = \int_0^T \varphi(t,S_{t-}) d[S]_t^c = \int_0^T dt\, \varphi(t,S_{t-}) S_{t-}^2 \delta_t^2. \tag{3.10}$$

Since φ is bounded and has compact support, in order to apply Fubini's theorem to

$$\mathbb{E}\left[\int_0^{+\infty} dK \int_0^T \varphi(t,K)\, dL_t^K\right],$$

it is sufficient to show that $\mathbb{E}\left[L_t^K\right] < \infty$ for $t \in [0,T]$. Rewriting equation (3.9) yields

$$\begin{aligned}\frac{1}{2}L_T^K &= (S_T - K)^+ - (S_0 - K)^+ - \int_0^T 1_{\{S_{t-}>K\}} dS_t \\ &- \sum_{0<t\leq T} \left[(S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t\right].\end{aligned}$$

Since $\hat{S}$ is a martingale, $\mathbb{E}[S_T] < \infty$, $\mathbb{E}\left[(S_T - K)^+\right] < \mathbb{E}\left[S_T\right]$ and $\mathbb{E}\left[\int_0^T 1_{\{S_{t-}>K\}} dS_t\right] < \infty$. As discussed above,

$$\mathbb{E}\left[\sum_{0<t\leq T} \left((S_t - K)^+ - (S_{t-} - K)^+ - 1_{\{S_{t-}>K\}}\Delta S_t\right)\right] < \infty,$$

yielding that $\mathbb{E}\left[L_T^K\right] < \infty$. Hence, one may take expectations in equation (3.10) to obtain

$$\begin{aligned}\mathbb{E}\left[\int_0^{+\infty} dK \int_0^T \varphi(t,K)\, dL_t^K\right] &= \mathbb{E}\left[\int_0^T \varphi(t,S_{t-}) S_{t-}^2 \delta_t^2 dt\right] \\ &= \int_0^T dt\, \mathbb{E}\left[\varphi(t,S_{t-}) S_{t-}^2 \delta_t^2\right] \\ &= \int_0^T dt\, \mathbb{E}\left[\mathbb{E}\left[\varphi(t,S_{t-}) S_{t-}^2 \delta_t^2 | S_{t-}\right]\right] \\ &= \mathbb{E}\left[\int_0^T dt\, \varphi(t,S_{t-}) S_{t-}^2 \sigma(t,S_{t-})^2\right] \\ &= \int_0^\infty \int_0^T \varphi(t,K) K^2 \sigma(t,K)^2 p_t^S(dK)\, dt \\ &= \int_0^T dt\, e^{\int_0^t r(s)\, ds} \int_0^\infty \varphi(t,K) K^2 \sigma(t,K)^2 \frac{\partial^2 C}{\partial K^2}(t,dK),\end{aligned}$$

where the last line is obtained by using (3.8). Using integration by parts,

$$
\begin{aligned}
& \int_0^\infty dK \int_0^T dt\, \varphi(t,K)\, \frac{\partial}{\partial t}\left[e^{\int_0^t r(s)\,ds}\, C(t,K) - (S_0 - K)^+\right] \\
= \ & \int_0^\infty dK \int_0^T dt\, \varphi(t,K)\, \frac{\partial}{\partial t}\left[e^{\int_0^t r(s)\,ds}\, C(t,K)\right] \\
= \ & \int_0^\infty dK \int_0^T dt\, \varphi(t,K)\, e^{\int_0^t r(s)\,ds}\left[\frac{\partial C}{\partial t}(t,K) + r(t)C(t,K)\right] \\
= \ & -\int_0^\infty dK \int_0^T dt\, \frac{\partial \varphi}{\partial t}(t,K)\, \left[e^{\int_0^t r(s)\,ds}\, C(t,K)\right],
\end{aligned}
$$

where derivatives are used in the sense of distributions. Gathering together all terms,

$$
\begin{aligned}
& \int_0^\infty dK \int_0^T dt\, \frac{\partial \varphi}{\partial t}(t,K)\, \left[e^{\int_0^t r(s)\,ds}\, C(t,K)\right] \\
= \ & \int_0^T dt \int_0^\infty dK\, \frac{\partial \varphi}{\partial t}(t,K) \int_0^t ds\, r(s)\, e^{\int_0^s r(u)\,du}[C(s,K) - K\frac{\partial C}{\partial K}(s,K)] \\
+ \ & \int_0^T dt \int_0^\infty \frac{1}{2}\frac{\partial \varphi}{\partial t}(t,K) \int_0^t dL_s^K \\
+ \ & \int_0^T dt \int_0^\infty dK\, \frac{\partial \varphi}{\partial t}(t,K) \int_0^t ds\, e^{\int_0^s r(u)\,du} \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(s,dy)\chi_{s,y}\left(\ln\left(\frac{K}{y}\right)\right) \\
= \ & -\int_0^T dt \int_0^\infty dK\, \varphi(t,K)\, r(t)\, e^{\int_0^t r(s)\,ds}[C(t,K) - K\frac{\partial C}{\partial K}(t,K)] \\
- \ & \int_0^T dt \int_0^\infty \frac{1}{2}\, \varphi(t,K)\, dL_t^K \\
- \ & \int_0^T dt \int_0^\infty dK\, \varphi(t,K)\, e^{\int_0^t r(s)\,ds} \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(t,dy)\chi_{t,y}\left(\ln\left(\frac{K}{y}\right)\right).
\end{aligned}
$$

So, for any $T > 0$ and any test function $\varphi \in C_0^\infty([0,T]\times]0,\infty[,\mathbb{R})$,

$$
\begin{aligned}
& \int_0^\infty dK \int_0^T dt\, \frac{\partial \varphi}{\partial t}(t,K)\ \left[e^{\int_0^t r(s)\,ds}\, C(t,K) \right] \\
= & \ -\int_0^\infty dK \int_0^T dt\, \varphi(t,K)\, e^{\int_0^t r(s)\,ds} \left[\frac{\partial C}{\partial t}(t,K) + r(t)C(t,K) \right] \\
= & \ -\int_0^T dt \int_0^\infty dK\, \varphi(t,K)\, r(t)\, e^{\int_0^t r(s)\,ds} [C(t,K) - K \frac{\partial C}{\partial K}(t,K)] \\
- & \ \int_0^T dt \int_0^\infty \frac{1}{2} e^{\int_0^t r(s)\,ds}\ \varphi(t,K)\, K^2 \sigma(t,K)^2 \frac{\partial^2 C}{\partial K^2}(t,dK) \\
- & \ \int_0^T dt \int_0^\infty dK\, \varphi(t,K)\, e^{\int_0^t r(s)\,ds}\ \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(t,dy) \chi_{t,y} \left(\ln \left(\frac{K}{y} \right) \right).
\end{aligned}
$$

Therefore, $C(.,.)$ is a solution of (3.5) in the sense of distributions. $\square$

3.1.3 Uniqueness of solutions of the forward PIDE

Theorem 3.1 shows that the call price $(T,K) \mapsto C_{t_0}(T,K)$ solves the forward PIDE (3.5). Uniqueness of the solution of such PIDEs has been shown using analytical methods [8, 47] under various types of conditions on the coefficients. We give below a direct proof of uniqueness for (3.5) using a probabilistic method, under explicit conditions which cover most examples of models used in finance.

Let $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ be the set of continuous functions defined on $[0,\infty[\times\mathbb{R}^+$ and vanishing at infinity for the supremum norm. Define, for $u \in \mathbb{R}, t \in [0,\infty[, z > 0$ the measure $n(t,du,z)$ by

$$
\begin{aligned}
n(t,[u,\infty[,z) &= -e^{-u} \frac{\partial}{\partial u} \left[\chi_{t,z}(u) \right], \quad u > 0\ ; \\
n(t,]-\infty,u],z) &= e^{-u} \frac{\partial}{\partial u} \left[\chi_{t,z}(u) \right], \quad u < 0.
\end{aligned}
\tag{3.11}
$$

Throughout this section, we make the following assumption:

Assumption 3.3.

$$
\forall T > 0, \forall B \in \mathcal{B}(\mathbb{R}) - \{0\}, \quad (t,z) \to \sigma(t,z), \quad (t,z) \to n(t,B,z)
$$

are continuous in $z \in \mathbb{R}^+$, uniformly in $t \in [0,T]$; continuous in t on $[0,T]$ uniformly in $z \in \mathbb{R}^+$ and

$$\exists K_1 > 0, \quad \forall (t,z) \in [0,T] \times \mathbb{R}^+, \quad |\sigma(t,z)| + \int_{\mathbb{R}} (1 \wedge |z|^2)\, n(t,du,z) \le K_1$$

Theorem 3.2. *Under Assumption 3.3, if*

$$\begin{cases} \text{either} \quad (i) \quad \forall R > 0 \ \forall t \in [0,T], \quad \inf_{0 \le z \le R} \sigma(t,z) > 0, \\ \text{or} \quad (ii) \quad \sigma(t,z) \equiv 0 \quad \text{and there exists} \quad \beta \in]0,2[, C > 0, K_2 > 0, \text{and a family} \\ n^\beta(t,du,z) \text{ of positive measures such that} \\ \forall (t,z) \in [0,T] \times \mathbb{R}^+, \quad n(t,du,z) = n^\beta(t,du,z) + 1_{\{|u| \le 1\}} \frac{C}{|u|^{1+\beta}}\, du, \\ \int \left(1 \wedge |u|^\beta\right)\, n^\beta(t,du,z) \le K_2, \quad \lim_{\epsilon \to 0} \sup_{z \in \mathbb{R}^+} \int_{|u| \le \epsilon} |u|^\beta\, n^\beta(t,du,z) = 0. \end{cases}$$

$$\text{and} \quad (iii) \quad \lim_{R \to \infty} \int_0^T \sup_{z \in \mathbb{R}^+} n\left(t, \{|u| \ge R\}, z\right)\, dt = 0,$$

then the call option price $(T,K) \mapsto C_{t_0}(T,K)$, as a function of maturity and strike, is the unique solution in the sense of distributions of the partial integro-differential equation (3.5) on $[t_0,\infty[\times]0,\infty[$ such that,

$$C_{t_0}(.,.) \in \mathcal{C}_0([t_0,T] \times \mathbb{R}^+),$$
$$\forall K > 0 \quad C_{t_0}(t_0,K) = (S_{t_0} - K)_+.$$

The proof uses the uniqueness of the solution of the forward Kolmogorov equation associated to a certain integro-differential operator. We start by recalling the following result:

Proposition 3.1. *Define, for $t \ge 0$ and $f \in C_0^\infty(\mathbb{R}^+)$, the integro-differential operator*

$$\begin{aligned} L_t f(x) &= r(t) x f'(x) + \frac{x^2 \sigma(t,x)^2}{2} f''(x) \\ &\quad + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1) f'(x)]\, n(t,dy,x). \end{aligned} \tag{3.12}$$

Under Assumption 3.3, if either conditions ((i) or (ii)) and (iii) of Theorem 3.2 hold, then for each x_0 in $\mathbb{R}^+$, there exists a unique family $(p_t(x_0, dy), t \geq 0)$ of positive bounded measures such that

$$\forall t \geq 0, \forall g \in \mathcal{C}_0^\infty(\mathbb{R}^+), \qquad \int_{\mathbb{R}} p_t(x_0, dy) g(y) = g(x_0) + \int_0^t \int_{\mathbb{R}} p_s(x_0, dy)\, L_s g(y)\, ds \tag{3.13}$$

$$p_0(x_0, .) = \epsilon_{x_0}, \tag{3.14}$$

where ϵ_{x_0} is the point mass at x_0. Furthermore, $p_t(x_0, .)$ is a probability measure on $[0, \infty[$.

Proof. Denote by $(X_t)_{t\in[0,T]}$ the canonical process on $D([0,T], \mathbb{R}_+)$. Under assumptions (i) (or (ii)) and (iii), the martingale problem for $((L_t)_{t\in[0,T]}, \mathcal{C}_0^\infty(\mathbb{R}^+))$ on $[0,T]$ is well-posed [76, Theorem 1]: for any $x_0 \in \mathbb{R}^+, t_0 \in [0, T[$, there exists a unique probability measure $\mathbb{Q}_{t_0,x_0}$ on $(D([0,T], \mathbb{R}^+), \mathcal{B}_T)$ such that $\mathbb{Q}\left(X_u = x_0,\, 0 \leq u \leq t_0\right) = 1$ and for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^+)$,

$$f(X_t) - f(x_0) - \int_{t_0}^t L_s f(X_s)\, ds$$

is a $(\mathbb{Q}_{t_0,x_0}, (\mathcal{B}_t)_{t\geq 0})$-martingale on $[0,T]$. Under $\mathbb{Q}_{t_0,x_0}$, X is a Markov process. Define the evolution operator $(Q_{t_0,t})_{t\in[0,T]}$ by

$$\forall f \in \mathcal{C}_b^0(\mathbb{R}^+), \quad Q_{t_0,t} f(x_0) = \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[f(X_t)\right]. \tag{3.15}$$

For $f \in \mathcal{C}_0^\infty(\mathbb{R}^+)$,

$$Q_{t_0,t} f(x_0) \;=\; f(x_0) + \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[\int_{t_0}^t L_s f(X_s)\, ds\right].$$

Given Assumption 3.3, $t \in [0,T] \mapsto \int_0^t L_s f(X_s)\, ds$ is uniformly bounded on $[0,T]$. Given Assumption 3.3, since X is right continuous, $s \in [0, T[\mapsto L_s f(X_s)$ is right-continuous up to a $\mathbb{Q}_{t_0,x_0}$-null set and

$$\lim_{t\downarrow t_0} \int_{t_0}^t L_s f(X_s)\, ds = 0 \quad \text{a.s.}$$

Applying the dominated convergence theorem yields

$$\lim_{t\downarrow t_0} \mathbb{E}^{\mathbb{Q}_{t_0,x_0}}\left[\int_{t_0}^t L_s f(X_s)\, ds\right] = 0, \qquad \text{so} \quad \lim_{t\downarrow t_0} Q_{t_0,t} f(x_0) = f(x_0),$$

implying that $t \in [0, T[\mapsto Q_{t_0,t}f(x_0)$ is right-continuous at t_0 for each $x_0 \in \mathbb{R}^+$ on $\mathcal{C}_0^\infty(\mathbb{R}^+)$. Hence the evolution operator $(Q_{t_0,t})_{t\in[t_0,T]}$ verifies the following continuity property:

$$\forall f \in \mathcal{C}_0^\infty(\mathbb{R}^+), \forall x \in \mathbb{R}^+, \quad \lim_{t\downarrow t_0} Q_{t_0,t}f(x) = f(x), \tag{3.16}$$

that is $(Q_{s,t}, 0 \leq s \leq t)$ is (time-inhomogeneous) semigroup strongly continuous on $\mathcal{C}_0^\infty(\mathbb{R}^+)$.

If $q_{t_0,t}(x_0, dy)$ denotes the law of X_t under $\mathbb{Q}_{t_0,x_0}$, the martingale property implies that $q_{t_0,t}(x_0, dy)$ satisfies

$$\forall g \in \mathcal{C}_0^\infty(\mathbb{R}^+), \quad \int_{\mathbb{R}^+} q_{t_0,t}(x_0, dy)g(y) = g(x_0) + \int_{t_0}^t \int_{\mathbb{R}^+} q_{t_0,s}(x_0, dy)L_s g(y)\, ds. \tag{3.17}$$

And also, the map

$$t \in [t_0, T[\mapsto \int_{\mathbb{R}^+} q_{t_0,t}(dy)f(y) \tag{3.18}$$

is right-continuous, for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^+), x_0 \in \mathbb{R}^+$.

$q_{t_0,t}$ is a solution of (3.13) (when $t_0 = 0$) with initial condition $q_{t_0,t_0}(dy) = \epsilon_{x_0}$. In particular, the measure $q_{t_0,t}$ has mass 1.

To show uniqueness of solutions of (3.13), we will rewrite (3.13) as the forward Kolmogorov equation associated with a *homogeneous* operator on space-time domain and use uniqueness results for the corresponding homogeneous equation.

1. Let $\mathcal{D}^0 \equiv \mathcal{C}^1([0, \infty[) \otimes \mathcal{C}_0^\infty(\mathbb{R}^+)$ be the tensor product of $\mathcal{C}^1([0, \infty[)$ and $\mathcal{C}_0^\infty(\mathbb{R}^+)$. Define the operator A on $\mathcal{D}^0$ by

   $$\forall f \in C_0^\infty(\mathbb{R}^+), \forall \gamma \in \mathcal{C}^1([0, \infty[), \quad A(f\gamma)(t, x) = \gamma(t)\mathcal{L}_t f(x) + f(x)\gamma'(t). \tag{3.19}$$

   [40, Theorem 7.1, Chapter 4] implies that for any $x_0 \in \mathbb{R}^+$, if $(X, \mathbb{Q}_{t_0,x_0})$ is a solution of the martingale problem for $\mathcal{L}$, then the law of $\eta_t = (t, X_t)$ under $\mathbb{Q}_{t_0,x_0}$ is a solution of the martingale problem for A: in particular for any $f \in \mathcal{C}_0^\infty(\mathbb{R}^+)$ and $\gamma \in \mathcal{C}^1([0, \infty[)$,

   $$\int q_{t_0,t}(x_0, dy)f(y)\gamma(t) = f(x_0)\gamma(0) + \int_{t_0}^t \int q_{t_0,s}(x_0, dy)A(f\gamma)(s, y)\, ds. \tag{3.20}$$

[40, Theorem 7.1, Chapter 4] implies also that if the law of $\eta_t = (t, X_t)$ is a solution of the martingale problem for A then the law of X is also a solution of the martingale problem for $\mathcal{L}$, namely: uniqueness holds for the martingale problem associated to the operator $\mathcal{L}$ on $\mathcal{C}_0^\infty(\mathbb{R}^+)$ if and only if uniqueness holds for the martingale problem associated to the martingale problem for A on $\mathcal{D}^0$.

Define, for $t \geq 0$ and $h \in \mathcal{C}_b^0([0,\infty[\times\mathbb{R}^+)$,

$$\forall (s,x) \in [0,\infty[\times\mathbb{R}^+, \quad \mathcal{U}_t h(s,x) = Q_{s,s+t}\left(h(t+s,.)\right)(x). \qquad (3.21)$$

The properties of $Q_{s,t}$ then imply that $(\mathcal{U}_t, t \geq 0)$ is a family of linear operators on $\mathcal{C}_b^0([0,\infty[\times\mathbb{R}^+)$ satisfying $\mathcal{U}_t\mathcal{U}_r = \mathcal{U}_{t+r}$ on $\mathcal{C}_b^0([0,\infty[\times\mathbb{R}^+)$ and $\mathcal{U}_t h \to h$ in as $t \downarrow 0$ on $\mathcal{D}^0$. $(\mathcal{U}_t, t \geq 0)$ is thus a semigroup on $\mathcal{C}_b^0([0,\infty[\times\mathbb{R}^+)$ satisfying a continuity property on $\mathcal{D}^0$:

$$\forall h \in \mathcal{D}^0, \quad \lim_{t \downarrow \epsilon} \mathcal{U}_t h(s,s) = \mathcal{U}_\epsilon h(s,x).$$

We intend to apply [40, Theorem 2.2, Chapter 4] to prove that $(\mathcal{U}_t, t \geq 0)$ generates a strongly continuous contraction on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ with infinitesimal generator given by the closure $\overline{A}$. First, one shall simply observe that $\mathcal{D}^0$ is dense in $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ implying that the domain of A is dense in $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ too. The well-posedness of the martingale problem for A implies that A satisfies the maximum principle. To conclude, it is sufficient to prove that $Im(\lambda - \overline{A})$ is dense in $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ for some λ or even better that $\mathcal{D}^0$ is included in $Im(\lambda - \overline{A})$. We recall that $Im(\lambda - \overline{A})$ denotes the image of $\mathcal{D}_0$ under the map $(\lambda - \overline{A})$.

2. Without loss of generality, let us put $t_0 = 0$ in the sequel. For $h \in \mathcal{D}^0$, the martingale property yields,

$$\begin{gathered} \forall 0 \leq \epsilon \leq t < T,\ \forall (s,x) \in [0,T] \times \mathbb{R}^+, \\ \mathcal{U}_t h - \mathcal{U}_\epsilon h = \int_\epsilon^t \mathcal{U}_u A h \, du, \end{gathered} \qquad (3.22)$$

which yields in turn

$$
\begin{aligned}
\int_\epsilon^T e^{-t}\mathcal{U}_t h\,dt &= \int_\epsilon^T e^{-t}\mathcal{U}_\epsilon h\,dt + \int_\epsilon^T e^{-t}\int_\epsilon^t \mathcal{U}_s A h\,ds\,dt \\
&= \mathcal{U}_\epsilon h\left[e^{-\epsilon} - e^{-T}\right] + \int_\epsilon^T ds\,\left(\int_s^T e^{-t}\,dt\right)\mathcal{U}_s A h \\
&= \mathcal{U}_\epsilon h\left[e^{-\epsilon} - e^{-T}\right] + \int_\epsilon^T ds\,\left[e^{-s} - e^{-T}\right]\mathcal{U}_s A h \\
&= e^{-\epsilon}\mathcal{U}_\epsilon h - e^{-T}\left[\mathcal{U}_\epsilon h + \int_\epsilon^T \mathcal{U}_s A h\,ds\right] \\
&+ \int_\epsilon^T ds\,e^{-s}\,\mathcal{U}_s A h.
\end{aligned}
$$

Using (3.22) and gathering all the terms together yields,

$$
\int_\epsilon^T e^{-t}\mathcal{U}_t h\,dt = e^{-\epsilon}\,\mathcal{U}_\epsilon h - e^{-T}\mathcal{U}_T h + \int_\epsilon^T ds\,e^{-s}\,\mathcal{U}_s A h. \tag{3.23}
$$

Let us focus on the quantity

$$
\int_\epsilon^T ds\,e^{-s}\,\mathcal{U}_s A h.
$$

Observing that,

$$
\frac{1}{\epsilon}\left[\mathcal{U}_{t+\epsilon}h - \mathcal{U}_t h\right] = \frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]\mathcal{U}_t h = \mathcal{U}_t\,\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h,
$$

since $h \in \mathrm{dom}(A)$, taking $\epsilon \to 0$ yields

$$
\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h \to Ah.
$$

Given Assumption 3.3, $Ah \in \mathcal{C}_b^0([0,\infty[\times\mathbb{R}^+)$ and the contraction property of $\mathcal{U}$ yields,

$$
\|\mathcal{U}_t\left(\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h - Ah\right)\| \leq \|\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h - Ah\|,
$$

where $\|.\|$ denotes the supremum norm on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$. Thus

$$
\lim_{\epsilon\to 0}\mathcal{U}_t\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]h = U_t A h
$$

Hence, the limit when $\epsilon \to 0$ of

$$\frac{1}{\epsilon}\left[\mathcal{U}_\epsilon - I\right]\mathcal{U}_t h$$

exists, implying that $\mathcal{U}_t h$ belongs to the domain of $\bar{A}$ for any $h \in \mathcal{D}^0$. Thus,

$$\int_\epsilon^T ds\, e^{-s}\,\mathcal{U}_s h$$

belongs to the domain of $\overline{A}$ and

$$\int_\epsilon^T ds\, e^{-s}\,\mathcal{U}_s A h = \overline{A}\int_\epsilon^T ds\, e^{-s}\,\mathcal{U}_s h.$$

Since $\mathcal{U}$ is a contraction semigroup, thus a contraction, and given the continuity property of $\mathcal{U}_t$ on the space $\mathcal{D}^0$, one may take $\epsilon \to 0$ and $T \to \infty$ in (3.23), leading to

$$\int_0^\infty e^{-t}\mathcal{U}_t h\, dt = \mathcal{U}_0 + \overline{A}\int_0^\infty ds\, e^{-s}\,\mathcal{U}_s h.$$

Thus,

$$\left(I - \overline{A}\right)\int_0^\infty ds\, e^{-s}\,\mathcal{U}_s h(s,x) = \mathcal{U}_0 h(s,x) = h(s,x),$$

yielding $h \in Im(I - \overline{A})$. We have shown that $(\mathcal{U}_t, t \geq 0)$ generates a strongly continuous contraction on $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ with infinitesimal generator $\overline{A}$ (see [40, Theorem 2.2, Chapter 4]). The Hille-Yosida theorem [40, Proposition 2.6, Chapter 1] then implies that for all $\lambda > 0$

$$Im(\lambda - \overline{A}) = \mathcal{C}_0([0,\infty[\times\mathbb{R}^+).$$

3. Now let $p_t(x_0, dy)$ be another solution of (3.13). First, considering equation (3.13) for the particular function $g(y) = 1$, yields

$$\forall t \geq 0 \qquad \int_0^\infty p_t(x_0, dy) = 1,$$

and $p_t(x_0, dy)$ has mass 1.

Then, an integration by parts implies that, for $(f,\gamma) \in \mathcal{C}_0^\infty(\mathbb{R}^+) \times \mathcal{C}_0^1([0,\infty)$,

$$\int_{\mathbb{R}^+} p_t(x_0,dy) f(y)\gamma(t) = f(x_0)\gamma(0) + \int_0^t \int_{\mathbb{R}^+} p_s(x_0,dy) A(f\gamma)(s,y)\, ds. \tag{3.24}$$

Define, for $h \in \mathcal{C}_0([0,\infty) \times \mathbb{R}^+)$,

$$\forall (t,x_0) \in [0,\infty[\times\mathbb{R}^+, \qquad \mathcal{P}_t h(0,x_0) \;=\; \int_{\mathbb{R}^+} p_t(x_0,dy) h(t,y).$$

Using (3.24) we have, for $(f,\gamma) \in \mathcal{C}_0^\infty(\mathbb{R}^+) \times \mathcal{C}_0^1([0,\infty[)$,

$$\forall \epsilon > 0 \quad \mathcal{P}_t(f\gamma) - \mathcal{P}_\epsilon(f\gamma) = \int_\epsilon^t \int_{\mathbb{R}^+} p_u(dy) A(f\gamma)(u,y)\, du = \int_\epsilon^t \mathcal{P}_u(A(f\gamma))\, du, \tag{3.25}$$

and by linearity, for any $h \in \mathcal{D}^0$,

$$\mathcal{P}_t h - \mathcal{P}_\epsilon h = \int_\epsilon^t \mathcal{P}_u A h\, du. \tag{3.26}$$

Multiplying by $e^{-\lambda t}$ and integrating with respect to t we obtain, for $\lambda > 0$,

$$\begin{aligned}
\lambda \int_0^\infty e^{-\lambda t}\, \mathcal{P}_t h(0,x_0)\, dt &= h(0,x_0) + \lambda \int_0^\infty e^{-\lambda t} \int_0^t \mathcal{P}_u(Ah)(0,x_0)\, du\, dt \\
&= h(0,x_0) + \lambda \int_0^\infty \left(\int_u^\infty e^{-\lambda t} dt \right) \mathcal{P}_u(Ah)(0,x_0)\, du \\
&= h(0,x_0) + \int_0^\infty e^{-\lambda u}\, \mathcal{P}_u(Ah)(0,x_0)\, du.
\end{aligned}$$

Similarly, we obtain for any $\lambda > 0$,

$$\lambda \int_0^\infty e^{-\lambda t}\, \mathcal{U}_t h(0,x_0)\, dt \;=\; h(0,x_0) + \int_0^\infty e^{-\lambda u}\, \mathcal{U}_u(Ah)(0,x_0)\, du.$$

Hence for any $h \in \mathcal{D}^0$, we have

$$\int_0^\infty e^{-\lambda t}\, \mathcal{U}_t(\lambda - A) h(0,x_0)\, dt = h(0,x_0) = \int_0^\infty e^{-\lambda t}\, \mathcal{P}_t(\lambda - A) h(0,x_0)\, dt. \tag{3.27}$$

Using the density of $Im(\lambda - A)$ in $Im(\lambda - \overline{A}) = \mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$, we get

$$\forall g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^+), \qquad \int_0^\infty e^{-\lambda t}\,\mathcal{U}_t g\,(0,x_0)\,dt = \int_0^\infty e^{-\lambda t}\,\mathcal{P}_t g(0,x_0)\,dt, \tag{3.28}$$

so the Laplace transform of $t \mapsto \mathcal{P}_t g\,(0,x_0)$ is uniquely determined.
Using (3.26), for any $h \in \mathcal{D}^0$, $t \mapsto \mathcal{P}_t h(0,x_0)$ is right-continuous:

$$\forall h \in \mathcal{D}^0, \qquad \lim_{t\downarrow\epsilon} \mathcal{P}_t h(0,x_0) = \mathcal{P}_\epsilon h(0,x_0).$$

Furthermore, the density of $\mathcal{D}_0$ in $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ implies the weak-continuity of $t \to \mathcal{P}_t g(0,x_0)$ for any $g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$. Indeed, let $g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$, there exists $(h_n)_{n\geq 0} \in \mathcal{D}_0$ such that

$$\lim_{n\to\infty} \|g - h_n\| = 0$$

Then equation (3.26) yields,

$$\begin{aligned}
& |\mathcal{P}_t g(0,x_0) - \mathcal{P}_\epsilon g(0,x_0)| \\
= \;& |\mathcal{P}_t(g-h_n)(0,x_0) + (\mathcal{P}_t - \mathcal{P}_\epsilon)\,h_n(0,x_0) + \mathcal{P}_\epsilon(g-h_n)(0,x_0)| \\
\leq \;& |\mathcal{P}_t(g-h_n)(0,x_0)| + |\,(\mathcal{P}_t - \mathcal{P}_\epsilon)\,h_n(0,x_0)| + |\mathcal{P}_\epsilon(g-h_n)(0,x_0)| \\
\leq \;& 2\,\|g-h_n\| + |\,(\mathcal{P}_t - \mathcal{P}_\epsilon)\,h_n(0,x_0)|
\end{aligned}$$

Using the right-continuity of $t \mapsto \mathcal{P}_t h_n(0,x_0)$ for any $n \geq 0$, taking $t \downarrow \epsilon$ then $n \to \infty$, yields

$$\lim_{t\downarrow\epsilon} \mathcal{P}_t g(0,x_0) = \mathcal{P}_\epsilon g(0,x_0).$$

Thus the two right-continuous functions $t \mapsto \mathcal{P}_t g(0,x_0)$ and $t \mapsto \mathcal{U}_t g(0,x_0)$ have the same Laplace transform by (3.28), which implies they are equal:

$$\forall g \in \mathcal{C}_0([0,\infty[\times\mathbb{R}^+), \qquad \int g(t,y) q_{0,t}(x_0,dy) = \int g(t,y) p_t(x_0,dy). \tag{3.29}$$

By [40, Proposition 4.4, Chapter 3], $\mathcal{C}_0([0,\infty[\times\mathbb{R}^+)$ is convergence determining, hence separating, allowing us to conclude that $p_t(x_0,dy) = q_{0,t}(x_0,dy)$.

$\square$

We can now study the uniqueness of the forward PIDE (3.5) and prove Theorem 3.2.

Proof. of Theorem 3.2. We start by decomposing L_t as $L_t = A_t + B_t$ where

$$A_t f(y) = r(t) y f'(y) + \frac{y^2 \sigma(t,y)^2}{2} f''(y), \quad \text{and}$$
$$B_t f(y) = \int_{\mathbb{R}} [f(ye^z) - f(y) - y(e^z - 1) f'(y)] n(t, dz, y).$$

Then, in the sense of distibutions, using the fact that $y\frac{\partial}{\partial y}(y-x)^+ = x1_{\{y>x\}} + (y-x)_+ = y\,1_{\{y>x\}}$ and $\frac{\partial^2}{\partial y^2}(y-x)^+ = \epsilon_x(y)$ where ϵ_x is a unit mass at x, we obtain

$$A_t (y-x)^+ = r(t) y\, 1_{\{y>x\}} + \frac{y^2 \sigma(t,y)^2}{2} \epsilon_x(y) \qquad \text{and}$$

$$\begin{aligned} B_T (y-x)^+ &= \int_{\mathbb{R}} [(ye^z - x)^+ - (y-x)^+ - (e^z - 1)\left(x 1_{\{y>x\}} + (y-x)^+\right)] n(t, dz, y) \\ &= \int_{\mathbb{R}} [(ye^z - x)^+ - e^z (y-x)^+ - x(e^z - 1) 1_{\{y>x\}}] n(t, dz, y). \end{aligned}$$

Using Lemma 3.1 for the measure $n(t, dz, y)$ and $\psi_{t,y}$ its exponential double tail,

$$B_t (y-x)^+ = y \psi_{t,y} \left(\ln \left(\frac{x}{y} \right) \right)$$

Hence, in the sense of distibutions, the following identity holds

$$L_t (y-x)^+ = r(t) \left(x 1_{\{y>x\}} + (y-x)_+ \right) + \frac{y^2 \sigma(t,y)^2}{2} \epsilon_x(y) + y \psi_{t,y} \left(\ln \left(\frac{x}{y} \right) \right). \tag{3.30}$$

Let $f : [t_0, \infty[\times]0, \infty[\mapsto \mathbb{R}$ be a solution in the sense of distributions of (3.5) with the initial condition : $f(0,x) = (S_0 - x)^+$. Integration by parts yields

$$\begin{aligned}
& \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy) L_t (y-x)^+ \\
= \; & \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy) \left(r(t)(x 1_{\{y>x\}} + (y-x)_+) + \frac{y^2\sigma(t,y)^2}{2}\epsilon_x(y) + y\psi_{t,y}\left(\ln\left(\frac{x}{y}\right)\right)\right) \\
= \; & -r(t)x \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy) 1_{\{y>x\}} + r(t)\int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy)(y-x)^+ \\
+ \; & \frac{x^2\sigma(t,x)^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy) y\psi_{t,y}\left(\ln\left(\frac{x}{y}\right)\right) \\
= \; & -r(t)x\frac{\partial f}{\partial x} + r(t)f(t,x) + \frac{x^2\sigma(t,x)^2}{2}\frac{\partial^2 f}{\partial x^2} + \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy)\, y\, \psi_{t,y}\left(\ln\left(\frac{x}{y}\right)\right).
\end{aligned}$$

Hence given (3.5), the following equality holds

$$\frac{\partial f}{\partial t}(t,x) = -r(t)f(t,x) + \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy) L_t(y-x)^+ \,, \tag{3.31}$$

or, equivalently, after integration with respect to time t

$$e^{\int_0^t r(s)\,ds} f(t,x) - f(0,x) = \int_0^t \int_0^\infty e^{\int_0^t r(s)\,ds}\frac{\partial^2 f}{\partial x^2}(t,dy) L_t(y-x)^+. \tag{3.32}$$

Integration by parts shows that

$$f(t,x) = \int_0^\infty \frac{\partial^2 f}{\partial x^2}(t,dy)(y-x)^+. \tag{3.33}$$

Hence (3.31) may be rewritten as

$$\int_0^\infty e^{\int_0^t r(s)\,ds}\frac{\partial^2 f}{\partial x^2}(t,dy)(y-x)^+ - (S_0-x)^+ = \int_0^t \int_0^\infty e^{\int_0^s r(u)\,du}\frac{\partial^2 f}{\partial x^2}(s,dy) L_s(y-x)^+\,ds. \tag{3.34}$$

Define $q_t(dy) \equiv e^{\int_0^t r(s)\,ds}\,\frac{\partial^2 f}{\partial x^2}(t,dy)$, we have $q_0(dy) = \epsilon_{S_0}(dy) = p_0(S_0,dy)$. For $g \in \mathcal{C}_0^\infty(]0,\infty[,\mathbb{R})$, integration by parts yields

$$g(y) = \int_0^\infty g''(z)(y-z)^+\,dz. \tag{3.35}$$

Replacing the above expression in $\int_{\mathbb{R}} g(y) q_t(dy)$ and using (3.34) we obtain

$$\begin{aligned}
&\int_0^\infty g(y) q_t(dy) = \int_0^\infty g(y)\, e^{\int_0^t r(s)\,ds} \frac{\partial^2 f}{\partial x^2}(t, dy) \\
= &\int_0^\infty g''(z) \int_0^\infty e^{\int_0^t r(s)\,ds} \frac{\partial^2 f}{\partial x^2}(t, dy)(y-z)^+\, dz \\
= &\int_0^\infty g''(z)(S_0 - z)^+\, dz + \int_0^\infty g''(z) \int_0^t \int_0^\infty e^{\int_0^s r(u)\,du} \frac{\partial^2 f}{\partial x^2}(s, dy) L_s (y-z)^+\, dz \\
= &\ g(S_0) + \int_0^t \int_0^\infty e^{\int_0^s r(u)\,du} \frac{\partial^2 f}{\partial x^2}(s, dy) L_s [\int_0^\infty g''(z)(y-z)^+\, dz] \\
= &\ g(S_0) + \int_0^t \int_0^\infty q_s(dy) L_s g(y)\, ds.
\end{aligned}$$

This is none other than equation (3.13). By uniqueness of the solution $p_t(S_0, dy)$ of (3.13) in Proposition 3.1,

$$e^{\int_0^t r(s)\,ds} \frac{\partial^2 f}{\partial x^2}(t, dy) = p_t(S_0, dy).$$

One may rewrite equation (3.32) as

$$f(t, x) = e^{-\int_0^t r(s)\,ds} \left(f(0, x) + \int_0^\infty p_t(S_0, dy) L_t (y - x)^+ \right),$$

showing that the solution of (3.5) with initial condition $f(0, x) = (S_0 - x)^+$ is unique. □

3.2 Examples

We now give various examples of pricing models for which Theorem 3.1 allows to retrieve or generalize previously known forms of forward pricing equations.

3.2.1 Itô processes

When (S_t) is an Itô process i.e. when the jump part is absent, the forward equation (3.5) reduces to the Dupire equation [34]. In this case our result reduces to the following:

Proposition 3.2 (Dupire equation)**.** *Consider the price process* (S_t) *whose dynamics under the pricing measure* $\mathbb{P}$ *is given by*

$$S_T = S_0 + \int_0^T r(t) S_t dt + \int_0^T S_t \delta_t dW_t.$$

Assume there exists a measurable function $\sigma : [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$ *such that*

$$\forall t \in t \in [t_0, T], \qquad \sigma(t, S_{t-}) = \sqrt{\mathbb{E}\left[\delta_t^2 | S_{t^-}\right]}. \tag{3.36}$$

If

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \delta_t^2 \, dt\right)\right] < \infty \qquad a.s. \tag{3.37}$$

the call option price (3.2) *is a solution (in the sense of distributions) of the partial differential equation*

$$\frac{\partial C_{t_0}}{\partial T}(T, K) = -r(T) K \frac{\partial C_{t_0}}{\partial K}(T, K) + \frac{K^2 \sigma(T, K)^2}{2} \frac{\partial^2 C_{t_0}}{\partial K^2}(T, K) \tag{3.38}$$

on $[t_0, \infty[\times]0, \infty[$ *with the initial condition:*

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

Notice in particular that this result does not require a non-degeneracy condition on the diffusion term.

Proof. It is sufficient to take $\mu \equiv 0$ in (3.1) then equivalently in (3.5). We leave the end of the proof to the reader. $\square$

3.2.2 Markovian jump-diffusion models

Another important particular case in the literature is the case of a Markov jump-diffusion driven by a Poisson random measure. Andersen and Andreasen [4] derived a forward PIDE in the situation where the jumps are driven by a compound Poisson process with time-homogeneous Gaussian jumps. We will now show here that Theorem 3.1 implies the PIDE derived in [4], given here in a more general context allowing for a time- and state-dependent Lévy measure, as well as infinite number of jumps per unit time ("infinite jump activity").

Proposition 3.3 (Forward PIDE for jump diffusion model)**.** *Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by*

$$S_t = S_0 + \int_0^T r(t) S_{t-} dt + \int_0^T S_{t-}\sigma(t, S_{t-}) dB_t + \int_0^T \int_{-\infty}^{+\infty} S_{t-}(e^y - 1)\tilde{N}(dt dy) \tag{3.39}$$

where B_t is a Brownian motion and N a Poisson random measure on $[0,T]\times\mathbb{R}$ with compensator $\nu(dz)\,dt$, $\tilde{N}$ the associated compensated random measure. Assume that

$$\sigma(.,.) \quad \text{is bounded} \qquad \text{and} \qquad \int_{\{|y|>1\}} e^{2y}\nu(dy) < \infty. \tag{3.40}$$

Then the call option price

$$C_{t_0}(T,K) = e^{-\int_{t_0}^T r(t)\,dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}]$$

is a solution (in the sense of distributions) of the PIDE

$$\begin{aligned}\frac{\partial C_{t_0}}{\partial T}(T,K) = &-r(T)K\frac{\partial C_{t_0}}{\partial K}(T,K) + \frac{K^2\sigma(T,K)^2}{2}\frac{\partial^2 C_{t_0}}{\partial K^2}(T,K) \\ &+ \int_{\mathbb{R}} \nu(dz)\, e^z \left[C_{t_0}(T, Ke^{-z}) - C_{t_0}(T,K) - K(e^{-z}-1)\frac{\partial C_{t_0}}{\partial K}\right]\end{aligned} \tag{3.41}$$

on $[t_0, \infty[\times]0, \infty[$ with the initial condition:

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+.$$

Proof. As in the proof of Theorem 3.1, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{\mathcal{F}_{t_0}}$ given $\mathcal{F}_{t_0}$, we may replace the conditional expectation in (3.2) by an expectation with respect to the marginal distribution $p_T^S(dy)$ of S_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0 = 0$ in the sequel, consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets and we denote $C_0(T,K) \equiv C(T,K)$ for simplicity.

Differentiating (3.2) in the sense of distributions with respect to K, we obtain:

$$\frac{\partial C}{\partial K}(T,K) = -e^{-\int_0^T r(t)\,dt}\int_K^{\infty} p_T^S(dy), \qquad \frac{\partial^2 C}{\partial K^2}(T,dy) = e^{-\int_0^T r(t)\,dt} p_T^S(dy).$$

In this particular case, $m(t,dz)\,dt \equiv \nu(dz)\,dt$ and ψ_t are simply given by:

$$\psi_t(z) \equiv \psi(z) = \begin{cases} \int_{-\infty}^{z} dx\; e^x \int_{-\infty}^{x} \nu(du) & z < 0 \\ \int_{z}^{+\infty} dx\; e^x \int_{x}^{\infty} \nu(du) & z > 0 \end{cases}$$

Then (3.4) yields

$$\chi_{t,S_{t-}}(z) = \mathbb{E}\left[\psi_t\left(z\right)|S_{t-}\right] = \psi(z).$$

Let us now focus on the term

$$\int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi\left(\ln\left(\frac{K}{y}\right)\right)$$

in (3.5). Applying Lemma 3.1 yields

$$\begin{aligned}
& \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi\left(\ln\left(\frac{K}{y}\right)\right) \\
= & \int_0^{\infty} e^{-\int_0^T r(t)\,dt} p_T^S(dy) \int_{\mathbb{R}} [(ye^z - K)^+ - e^z(y-K)^+ - K(e^z-1)1_{\{y>K\}}]\nu(dz) \\
= & \int_{\mathbb{R}} e^z \int_0^{\infty} e^{-\int_0^T r(t)\,dt}\, p_T^S(dy)[(y-Ke^{-z})^+ - (y-K)^+ - K(1-e^{-z})1_{\{y>K\}}]\,\nu(dz) \\
= & \int_{\mathbb{R}} e^z \left[C(T,Ke^{-z}) - C(T,K) - K(e^{-z}-1)\frac{\partial C}{\partial K}\right]\nu(dz). \qquad (3.42)
\end{aligned}$$

This ends the proof. $\square$

3.2.3 Pure jump processes

For price processes with no Brownian component, Assumption (H) reduces to

$$\forall T > 0, \quad \mathbb{E}\left[\exp\left(\int_0^T dt \int (e^y - 1)^2 m(t,\, dy)\right)\right] < \infty.$$

Assume there exists a measurable function $\chi : [t_0,T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$ such that for all $t \in [t_0,T]$ and for all $z \in \mathbb{R}$:

$$\chi_{t,S_{t-}}(z) = \mathbb{E}\left[\psi_t\left(z\right)|S_{t-}\right], \qquad (3.43)$$

with

$$\psi_T(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} m(T,du), & z < 0\ ; \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} m(T,du), & z > 0, \end{cases}$$

then, the forward equation for call option becomes

$$\frac{\partial C}{\partial T} + r(T)K\frac{\partial C}{\partial K} = \int_0^{+\infty} y\frac{\partial^2 C}{\partial K^2}(T,dy)\,\chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right). \tag{3.44}$$

It is convenient to use the change of variable: $v = \ln y, k = \ln K$. Define $c(k,T) = C(e^k,T)$. Then one can write this PIDE as

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} e^{2(v-k)}\left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)(T,dv)\,\chi_{T,v}(k-v). \tag{3.45}$$

In the case, considered in [25], where the Lévy density m_Y has a deterministic separable form

$$m_Y(t,dz,y)\,dt = \alpha(y,t)\,k(z)\,dz\,dt, \tag{3.46}$$

Equation (3.45) allows us to recover[1] equation (14) in [25]

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = \int_{-\infty}^{+\infty} \kappa(k-v)e^{2(v-k)}\alpha(e^v,T)\left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)(T,dv)$$

where κ is defined as the exponential double tail of $k(u)\,du$, i.e.

$$\kappa(z) = \begin{cases} \int_{-\infty}^{z} dx\ e^x \int_{-\infty}^{x} k(u)\,du & z < 0\ ; \\ \int_{z}^{+\infty} dx\ e^x \int_{x}^{\infty} k(u)\,du & z > 0. \end{cases}$$

The right hand side can be written as a convolution of distributions:

$$\frac{\partial c}{\partial T} + r(T)\frac{\partial c}{\partial k} = [a_T(.)\ \left(\frac{\partial^2 c}{\partial k^2} - \frac{\partial c}{\partial k}\right)] * g \qquad \text{where} \tag{3.47}$$

$$g(u) = e^{-2u}\kappa(u) \qquad a_T(u) = \alpha(e^u,T). \tag{3.48}$$

Therefore, knowing $c(.,.)$ and given $\kappa(.)$ we can recover a_T hence $\alpha(.,.)$. As noted by Carr et al. [25], this equation is analogous to the Dupire formula for diffusions: it enables to "invert" the structure of the jumps–represented by α– from the cross-section of option prices. Note that, like the Dupire formula, this inversion involves a double deconvolution/differentiation of c which illustrates the ill-posedness of the inverse problem.

[1]Note however that the equation given in [25] does not seem to be correct: it involves the double tail of $k(z)\,dz$ instead of the exponential double tail.

3.2.4 Time changed Lévy processes

Time changed Lévy processes were proposed in [24] in the context of option pricing. Consider the price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by

$$S_t \equiv e^{\int_0^t r(u)\,du} X_t \qquad X_t = \exp\left(L_{\Theta_t}\right) \qquad \Theta_t = \int_0^t \theta_s ds \tag{3.49}$$

where L_t is a Lévy process with characteristic triplet (b, σ^2, ν), N its jump measure and (θ_t) is a locally bounded positive semimartingale. X is a $\mathbb{P}$-martingale if

$$b + \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^z - 1 - z\,1_{\{|z|\leq 1\}})\nu(dy) = 0. \tag{3.50}$$

Define the value $C_{t_0}(T, K)$ at t_0 of the call option with expiry $T > t_0$ and strike $K > 0$ as

$$C_{t_0}(T, K) = e^{-\int_0^T r(t)\,dt} E^{\mathbb{P}}[\max(S_T - K, 0)|\mathcal{F}_{t_0}]. \tag{3.51}$$

Proposition 3.4. *Assume there exists a measurable function* $\alpha : [0, T]\times\mathbb{R} \mapsto \mathbb{R}$ *such that*

$$\alpha(t, X_{t-}) = E[\theta_t | X_{t-}], \tag{3.52}$$

and let χ *be the exponential double tail of* ν*, defined as*

$$\chi(z) = \begin{cases} \int_{-\infty}^{z} dx\; e^x \int_{-\infty}^{x} \nu(du), & z < 0\ ; \\ \int_{z}^{+\infty} dx\; e^x \int_{x}^{\infty} \nu(du), & z > 0. \end{cases} \tag{3.53}$$

If $\beta = \frac{1}{2}\sigma^2 + \int_{\mathbb{R}} (e^y - 1)^2 \nu(dy) < \infty$ *and*

$$\mathbb{E}\left[\exp\left(\beta\Theta_T\right)\right] < \infty, \tag{3.54}$$

then the call option price $C_{t_0} : (T, K) \mapsto C_{t_0}(T, K)$ *at date* t_0*, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation*

$$\begin{aligned} \frac{\partial C}{\partial T}(T, K) = -r\alpha(T, K) K \frac{\partial C}{\partial K}(T, K) + \frac{K^2\alpha(T, K)\sigma^2}{2}\frac{\partial^2 C}{\partial K^2}(T, K) \\ + \int_0^{+\infty} y \frac{\partial^2 C}{\partial K^2}(T, dy)\, \alpha(T, y)\, \chi\left(\ln\left(\frac{K}{y}\right)\right) \end{aligned} \tag{3.55}$$

on $[t, \infty[\times]0, \infty[$ *with the initial condition:* $\forall K > 0 \quad C_{t_0}(t_0, K) = (S_{t_0} - K)_+$.

Proof. Using Lemma 2.1, (L_{Θ_t}) writes

$$
\begin{aligned}
L_{\Theta_t} &= L_0 + \int_0^t \sigma\sqrt{\theta_s} dB_s + \int_0^t b\theta_s ds \\
&+ \int_0^t \theta_s \int_{|z|\le 1} z\tilde{N}(ds\,dz) + \int_0^t \int_{\{|z|>1\}} zN(ds\,dz)
\end{aligned}
$$

where N is an integer-valued random measure with compensator $\theta_t\nu(dz)\,dt$, $\tilde{N}$ its compensated random measure. Applying the Itô formula yields

$$
\begin{aligned}
X_t &= X_0 + \int_0^t X_{s-} dL_{T_s} + \frac{1}{2}\int_0^t X_{s-}\sigma^2\theta_s\,ds + \sum_{s\le t}(X_s - X_{s-} - X_{s-}\Delta L_{T_s}) \\
&= X_0 + \int_0^t X_{s-}\left[b\theta_s + \frac{1}{2}\sigma^2\theta_s\right] ds + \int_0^t X_{s-}\sigma\sqrt{\theta_s}\,dB_s \\
&+ \int_0^t X_{s-}\theta_s \int_{\{|z|\le 1\}} z\tilde{N}(ds\,dz) + \int_0^t X_{s-}\theta_s \int_{\{|z|>1\}} zN(ds\,dz) \\
&+ \int_0^t \int_{\mathbb{R}} X_{s-}(e^z - 1 - z)N(ds\,dz)
\end{aligned}
$$

Under our assumptions, $\int (e^z - 1 - z\,1_{\{|z|\le 1\}})\nu(dz) < \infty$, hence:

$$
\begin{aligned}
X_t &= X_0 + \int_0^t X_{s-}\left[b\theta_s + \frac{1}{2}\sigma^2\theta_s + \int_{\mathbb{R}}(e^z - 1 - z\,1_{\{|z|\le 1\}})\theta_s\nu(dz)\right] ds \\
&+ \int_0^t X_{s-}\sigma\sqrt{\theta}\,dB_s + \int_0^t \int_{\mathbb{R}} X_{s-}\theta_s(e^z - 1)\tilde{N}(ds\,dz) \\
&= X_0 + \int_0^t X_{s-}\sigma\sqrt{\theta_s}\,dB_s + \int_0^t \int_{\mathbb{R}} X_{s-}(e^z - 1)\tilde{N}(ds\,dz)
\end{aligned}
$$

and (S_t) may be expressed as

$$
S_t = S_0 + \int_0^t S_{s-}r(s)\,ds + \int_0^t S_{s-}\sigma\sqrt{\theta_s}\,dB_s + \int_0^t \int_{\mathbb{R}} S_{s-}(e^z - 1)\tilde{N}(ds\,dz).
$$

Assumption (3.54) implies that S fulfills Assumption (H) of Theorem 3.1 and (S_t) is now in the suitable form (3.1) to apply Theorem 3.1, which yields the result. $\square$

3.2.5 Index options in a multivariate jump-diffusion model

Consider a multivariate model with d assets

$$S_T^i = S_0^i + \int_0^T r(t) S_{t^-}^i dt + \int_0^T S_{t^-} \delta_t^i dW_t^i + \int_0^T \int_{\mathbb{R}^d} S_{t^-}^i (e^{y_i} - 1) \tilde{N}(dt\, dy)$$

where δ^i is an adapted process taking values in $\mathbb{R}$ representing the volatility of asset i, W is a d-dimensional Wiener process, N is a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with compensator $\nu(dy)\, dt$, $\tilde{N}$ denotes its compensated random measure. The Wiener processes W^i are correlated

$$\forall 1 \leq (i,j) \leq d, \langle W^i, W^j \rangle_t = \rho_{i,j} t,$$

with $\rho_{ij} > 0$ and $\rho_{ii} = 1$. An index is defined as a weighted sum of asset prices

$$I_t = \sum_{i=1}^d w_i S_t^i, \quad w_i > 0, \quad \sum_1^d w_i = 1, \quad d \geq 2.$$

The value $C_{t_0}(T,K)$ at time t_0 of an index call option with expiry $T > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(T,K) = e^{-\int_{t_0}^T r(t)\, dt} E^{\mathbb{P}}[\max(I_T - K, 0) | \mathcal{F}_{t_0}]. \tag{3.56}$$

The following result is a generalization of the forward PIDE studied by Avellaneda et al. [5] for the diffusion case:

Theorem 3.3. *Forward PIDE for index options. Assume*

$$\begin{cases} \forall T > 0 \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^T \|\delta_t\|^2\, dt\right)\right] < \infty \\ \int_{\mathbb{R}^d} (\|y\| \wedge \|y\|^2)\, \nu(dy) < \infty \quad a.s. \end{cases} \tag{3.57}$$

Define

$$\eta_t(z) = \begin{cases} \int_{-\infty}^z dx\ e^x \int_{\mathbb{R}^d} 1_{\ln\left(\frac{\sum_{1 \leq i \leq d-1} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right) \leq x} \nu(dy) \quad z < 0 \\ \int_z^\infty dx\ e^x \int_{\mathbb{R}^d} 1_{\ln\left(\frac{\sum_{1 \leq i \leq d-1} w_i S_{t-}^i e^{y_i}}{I_{t-}}\right) \geq x} \nu(dy) \quad z > 0 \end{cases} \tag{3.58}$$

and assume there exists measurable functions $\sigma : [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$, $\chi : [t_0, T] \times \mathbb{R}^+ - \{0\} \mapsto \mathbb{R}^+$ *such that for all* $t \in [t_0, T]$ *and for all* $z \in \mathbb{R}$*:*

$$\begin{cases} \sigma(t, I_{t-}) & = \frac{1}{z} \sqrt{\mathbb{E}\left[\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right) | I_{t^-}\right]} \quad a.s., \\ \chi_{t,I_{t-}}(z) & = \mathbb{E}\left[\eta_t\left(z\right) | I_{t-}\right] \quad a.s. \end{cases} \tag{3.59}$$

Then the index call price $(T, K) \mapsto C_{t_0}(T, K)$*, as a function of maturity and strike, is a solution (in the sense of distributions) of the partial integro-differential equation*

$$\frac{\partial C_{t_0}}{\partial T} = -r(T)K\frac{\partial Ct_0}{\partial K} + \frac{\sigma(T,K)^2}{2}\,\frac{\partial^2 Ct_0}{\partial K^2} + \int_0^{+\infty} y \frac{\partial^2 Ct_0}{\partial K^2}(T, dy)\, \chi_{T,y}\left(\ln\left(\frac{K}{y}\right)\right) \tag{3.60}$$

on $[t_0, \infty[\times]0, \infty[$ *with the initial condition:*

$$\forall K > 0, \quad C_{t_0}(t_0, K) = (I_{t_0} - K)_+.$$

Proof. $(B_t)_{t\geq 0}$ defined by

$$dB_t = \frac{\sum_{i=1}^d w_i S_{t-}^i \delta_t^i dW_t^i}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{1/2}}$$

is a continuous local martingale with quadratic variation t: by Lévy's theorem, B is a Brownian motion. Hence I may be decomposed as

$$\begin{aligned} I_T = {} & \sum_{i=1}^d w_i S_0^i + \int_0^T r(t) I_{t-}\, dt + \int_0^T \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j\right)^{\frac{1}{2}} dB_t \\ & + \int_0^T \int_{\mathbb{R}^d} \sum_{i=1}^d w_i S_{t-}^i (e^{y_i} - 1) \tilde{N}(dt\, dy) \end{aligned} \tag{3.61}$$

The essential part of the proof consists in rewriting (I_t) in the suitable form

(3.1) to apply Theorem 3.1. Applying the Itô formula to $\ln(I_T)$ yields

$$\begin{aligned}
\ln\frac{I_T}{I_0} &= \int_0^T \Big[r(t) - \frac{1}{2I_{t-}^2}\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S_{t-}^iS_{t-}^j \\
&\quad - \int \big(\frac{\sum_{1\le i\le d} w_iS_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\big(\frac{\sum_{1\le i\le d} w_iS_{t-}^i e^{y_i}}{I_{t-}}\big)\big)\nu(dy)\Big]\,dt \\
&+ \int_0^T \frac{1}{I_{t-}}\left(\sum_{i,j=1}^d w_iw_j\rho_{ij}\,\delta_t^i\delta_t^j\,S_{t-}^iS_{t-}^j\right)^{\frac{1}{2}} dB_t + \int_0^T\int \ln\left(\frac{\sum_{1\le i\le d} w_iS_{t-}^i e^{y_i}}{I_{t-}}\right)\tilde{N}(dt\,dy).
\end{aligned}$$

Using the concavity property of the logarithm,

$$\ln\left(\frac{\sum_{1\le i\le d} w_iS_{t-}^i\,e^{y_i}}{I_{t-}}\right) \ge \sum_{1\le i\le d}\frac{w_iS_{t-}^i}{I_{t-}}\,y_i \ge -\|y\|$$

and

$$\ln\left(\frac{\sum_{1\le i\le d} w_iS_{t-}^i\,e^{y_i}}{I_{t-}}\right) \le \ln\left(\max_{1\le i\le d} e^{y_i}\right) \le \max_{1\le i\le d} y_i.$$

Thus,

$$\left|\ln\left(\frac{\sum_{1\le i\le d} w_iS_{t-}^i\,e^{y_i}}{I_{t-}}\right)\right| \le \|y\|.$$

Hence,

$$\int e^{2\left|\ln\left(\frac{\sum_{1\le i\le d} w_iS_{t-}^i e^{y_i}}{I_{t-}}\right)\right|} \wedge \left(\ln\left(\frac{\sum_{1\le i\le d} w_iS_{t-}^i e^{y_i}}{I_{t-}}\right)\right)^2 \nu(dy) < \infty \quad a.s. \tag{3.62}$$

Similarly, (3.57) implies that $\int (e^{y_i} - 1 - y_i)\,\nu(dy) < \infty$ so $\ln(S_T^i)$ may be expressed as

$$\begin{aligned}
\ln(S_T^i) &= \ln(S_0^i) + \int_0^T \left(r(t) - \frac{1}{2}(\delta_t^i)^2 - \int (e^{y_i} - 1 - y_i)\,\nu(dy)\right)dt \\
&+ \int_0^T \delta_t^i\,dW_t^i + \int_0^T\int y_i\,\tilde{N}(dt\,dy)
\end{aligned}$$

Define the d-dimensional martingale $W_t = (W_t^1, \cdots, W_t^{d-1}, B_t)$. For $1 \leq i, j \leq d-1$ we have

$$\langle W^i, W^j \rangle_t = \rho_{i,j} t \quad \text{and} \quad \langle W^i, B \rangle_t = \frac{\sum_{j=1}^d w_j \rho_{ij} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{1/2}} \, t.$$

Define

$$\Theta_t = \begin{pmatrix} 1 & \cdots & \rho_{1,d-1} & \frac{\sum_{j=1}^d w_j \rho_{1j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{1/2}} \\ \vdots & \ddots & \vdots & \vdots \\ \rho_{d-1,1} & \cdots & 1 & \frac{\sum_{j=1}^d w_j \rho_{d-1,j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{1/2}} \\ \frac{\sum_{j=1}^d w_j \rho_{1,j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{1/2}} & \cdots & \frac{\sum_{j=1}^d w_j \rho_{d-1,j} S_{t-}^j \delta_t^j}{\left(\sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{1/2}} & 1 \end{pmatrix}$$

There exists a standard Brownian motion (Z_t) such that $W_t = AZ_t$ where A is a $d \times d$ matrix verifying $\Theta = {}^t A \, A$. Define $X_T \equiv \left(\ln(S_T^1), \cdots, \ln(S_T^{d-1}), \ln(I_T) \right)$;

$$\delta = \begin{pmatrix} \delta_t^1 & \cdots & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & \delta_t^{d-1} & 0 \\ 0 & \cdots & 0 & \frac{1}{I_{t-}} \left(\sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}} \end{pmatrix},$$

$$\beta_t = \begin{pmatrix} r(t) - \frac{1}{2}(\delta_t^1)^2 - \int (e^{y_1} - 1 - y_1) \, \nu(dy) \\ \vdots \\ r(t) - \frac{1}{2}(\delta_t^{d-1})^2 - \int (e^{y_{d-1}} - 1 - y_{d-1}) \, \nu(dy) \\ r(t) - \frac{1}{2I_{t-}^2} \sum_{i,j=1}^d w_i w_j \rho_{ij} \, \delta_t^i \delta_t^j \, S_{t-}^i S_{t-}^j - \int \left(\frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \right) \nu(dy) \end{pmatrix},$$

$$\text{and} \qquad \psi_t(y) = \begin{pmatrix} y_1 \\ \vdots \\ y_{d-1} \\ \ln\left(\frac{\sum_{1 \leq i \leq d} w_i S_{t-}^i e^{y_i}}{I_{t-}} \right) \end{pmatrix}.$$

Then X_T may be expressed as

$$X_T = X_0 + \int_0^T \beta_t \, dt + \int_0^T \delta_t A \, dZ_t + \int_0^T \int_{\mathbb{R}^d} \psi_t(y) \, \tilde{N}(dt \, dy) \tag{3.63}$$

The predictable function ϕ_t defined, for $t \in [0,T], y \in \psi_t(\mathbb{R}^d)$, by

$$\phi_t(y) = \left(y_1, \cdots, y_{d-1}, \ln\left(\frac{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S^i_{t-} e^{y_i}}{w_d S^d_{t-}} \right) \right)$$

is the left inverse of ψ_t: $\phi_t(\omega, \psi_t(\omega, y)) = y$. Observe that $\psi_t(.,0) = 0$, ϕ is predictable, and $\phi_t(\omega,.)$ is differentiable on $Im(\psi_t)$ with Jacobian matrix $\nabla_y \phi_t(y)$ given by

$$(\nabla_y \phi_t(y)) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \cdots & 1 & 0 \\ \frac{-e^{y_1} w_1 S^1_{t-}}{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S^i_{t-} e^{y_i}} & \cdots & \frac{-e^{y_{d-1}} w_{d-1} S^{d-1}_{t-}}{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S^i_{t-} e^{y_i}} & \frac{e^{y_d} I_{t-}}{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S^i_{t-} e^{y_i}} \end{pmatrix}$$

so (ψ, ν) satisfies the assumptions of Lemma 2.1: using Assumption (3.57), for all $T \ge t \ge 0$,

$$\begin{aligned} & \mathbb{E}\left[\int_0^T \int_{\mathbb{R}^d} (1 \wedge \|\psi_t(.,y)\|^2) \, \nu(dy) \, dt \right] \\ = & \int_0^T \int_{\mathbb{R}^d} 1 \wedge \left(y_1^2 + \cdots + y_{d-1}^2 + \ln\left(\frac{\sum_{1 \le i \le d} w_i S^i_{t-} e^{y_i}}{I_{t-}} \right)^2 \right) \nu(dy) \, dt \\ \le & \int_0^T \int_{\mathbb{R}^d} 1 \wedge (2\|y\|^2) \, \nu(dy) \, dt < \infty. \end{aligned}$$

Define ν_ϕ, the image of ν by ϕ by

$$\nu_\phi(\omega, t, B) = \nu(\phi_t(\omega, B)) \quad \text{for} \quad B \subset \psi_t(\mathbb{R}^d). \tag{3.64}$$

Applying Lemma 2.1, X_T may be expressed as

$$X_T = X_0 + \int_0^T \beta_t \, dt + \int_0^T \delta_t A \, dZ_t + \int_0^T \int y \, \tilde{M}(dt \, dy)$$

where M is an integer-valued random measure (resp. $\tilde{M}$ its compensated random measure) with compensator

$$\mu(\omega; dt\ dy) = m(t, dy; \omega)\, dt,$$

defined via its density

$$\frac{d\mu}{d\nu_\phi}(\omega, t, y) = 1_{\{\psi_t(\mathbb{R}^d)\}}(y)\, |\det \nabla_y \phi_t|(y) = 1_{\{\psi_t(\mathbb{R}^d)\}}(y) \left| \frac{e^{y_d} I_{t-}}{e^{y_d} I_{t-} - \sum_{1 \le i \le d-1} w_i S^i_{t-} e^{y_i}} \right|$$

with respect to ν_ϕ. Considering now the d-th component of X_T, one obtains the semimartingale decomposition of $\ln(I_t)$:

$$\begin{aligned}
&\ln(I_T) - \ln(I_0) \\
= &\int_0^T \Bigg(r(t) - \frac{1}{2 I_{t-}^2} \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S^i_{t-} S^j_{t-} \right) \\
&\qquad - \int \left(\frac{\sum_{1 \le i \le d} w_i S^i_{t-} e^{y_i}}{I_{t-}} - 1 - \ln\left(\frac{\sum_{1 \le i \le d} w_i S^i_{t-} e^{y_i}}{I_{t-}} \right) \right) \nu(dy) \Bigg)\, dt \\
+ &\int_0^T \frac{1}{I_{t-}} \left(\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S^i_{t-} S^j_{t-} \right)^{\frac{1}{2}} dB_t + \int_0^T \int y\, \tilde{K}(dt\, dy)
\end{aligned}$$

where K is an integer-valued random measure on $[0, T] \times \mathbb{R}$ with compensator $k(t, dy)\, dt$ where

$$\begin{aligned}
k(t, B) &= \int_{\mathbb{R}^{d-1} \times B} \mu(t, dy) = \int_{\mathbb{R}^{d-1} \times B} 1_{\{\psi_t(\mathbb{R}^d)\}}(y)\, |\det \nabla_y \phi_t|(y)\, \nu_\phi(t, dy) \\
&= \int_{\left(\mathbb{R}^{d-1} \times B\right) \cap \psi_t(\mathbb{R}^d)} |\det \nabla_y \phi_t|(\psi_t(y))\, \nu(dy) \\
&= \int_{\{y \in \mathbb{R}^d - \{0\}, \ln\left(\frac{\sum_{1 \le i \le d-1} w_i S^i_{t-} e^{y_i}}{I_{t-}}\right) \in B\}} \nu(dy) \quad \text{for} \quad B \in \mathcal{B}(\mathbb{R} - \{0\}).
\end{aligned}$$

In particular, the exponential double tail of $k(t, dy)$ which we denote $\eta_t(z)$

$$\eta_t(z) = \begin{cases} \int_{-\infty}^z dx\ e^x k(t,]-\infty, x]), & z < 0\ ; \\ \int_z^{+\infty} dx\ e^x k(t, [x, \infty[), & z > 0, \end{cases}$$

is given by (3.58). So finally I_T may be expressed as

$$
\begin{aligned}
I_T &= I_0 + \int_0^T r(t) I_{t-}\, dt + \int_0^T \left(\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j \right)^{\frac{1}{2}} dB_t \\
&+ \int_0^T \int_{\mathbb{R}^d} (e^y - 1)\, I_{t-} \tilde{K}(dt\, dy).
\end{aligned}
$$

The normalized volatility of I_t satisfies, for $t \in [0, T]$,

$$
\frac{\sum_{i,j=1}^{d} w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j}{I_{t-}^2} \leq \sum_{i,j=1}^{d} \rho_{ij}\, \delta_t^i \delta_t^j.
$$

Furthermore, clearly,

$$
\sum_{1 \leq i \leq d} \frac{w_i S_{t-}^i}{I_{t-}} e^{z_i} - 1 \leq e^{\|z\|} - 1,
$$

and the convexity property of the exponential yields

$$
\begin{aligned}
e^{\|z\|} + \sum_{1 \leq i \leq d} \frac{w_i S_{t-}^i}{I_{t-}} e^{z_i} &\geq e^{\|z\|} + \exp\left(\sum_{1 \leq i \leq d} \frac{w_i S_{t-}^i}{I_{t-}} z_i \right) \\
&= e^{\|z\|} + e^{\alpha_t . z} \geq e^{\|z\|} + e^{-\|\alpha_t\| \|z\|} \geq e^{\|z\|} + e^{-\|z\|} \geq 2,
\end{aligned}
$$

where

$$
\alpha_t = \left(\frac{w_1 S_{t-}^1}{I_{t-}}, \cdots, \frac{w_d S_{t-}^d}{I_{t-}} \right).
$$

Hence

$$
e^{\|z\|} - 1 \geq 1 - \sum_{1 \leq i \leq d} \frac{w_i S_{t-}^i}{I_{t-}} e^{z_i},
$$

and

$$
\left(\sum_{1 \leq i \leq d} \frac{w_i S_{t-}^i}{I_{t-}} e^{z_i} - 1 \right)^2 \leq \left(e^{\|z\|} - 1 \right)^2.
$$

Hence

$$
\begin{aligned}
& \frac{1}{2}\int_0^T \frac{\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j}{I_{t-}^2}\, dt + \int_0^T \int (e^y-1)^2 k(t,\, dy)\, dt \\
= \; & \frac{1}{2}\int_0^T \frac{\sum_{i,j=1}^d w_i w_j \rho_{ij}\, \delta_t^i \delta_t^j\, S_{t-}^i S_{t-}^j}{I_{t-}^2}\, dt \\
+ \; & \int_0^T \int_{\mathbb{R}^d} \left(\frac{\sum_{1\le i\le d-1} w_i S_{t-}^i e^{y_i} + w_d S_{t-}^d e^{y}}{I_{t-}} - 1 \right)^2 \nu(dy_1, \cdots, dy_{d-1}, dy)\, dt \\
\le \; & \frac{1}{2}\sum_{i,j=1}^d \rho_{ij}\, \delta_t^i \delta_t^j + \int_0^T \int_{\mathbb{R}^d} (e^{\|z\|}-1)^2 \nu(dz_1, \cdots, dz_d)\, dt.
\end{aligned}
$$

Using assumptions (3.57), the last inequality implies that I_t satisfies (H). Hence Theorem 3.1 can now be applied to I, which yields the result. □

3.2.6 Forward equations for CDO pricing

Portfolio credit derivatives such as CDOs or index default swaps are derivatives whose payoff depends on the total loss L_t due to defaults in a reference portfolio of obligors. Reduced-form top-down models of portfolio default risk [45, 48, 85, 28, 86] represent the default losses of a portfolio as a *marked point process* $(L_t)_{t\ge 0}$ where the jump times represents credit events in the portfolio and the jump sizes ΔL_t represent the portfolio loss upon a default event. Marked point processes with random intensities are increasingly used as ingredients in such models [45, 48, 73, 85, 86]. In all such models the loss process (represented as a fraction of the portfolio notional) may be represented as

$$L_t = \int_0^t \int_0^1 x\, M(ds\, dx),$$

where $M(dt\, dx)$ is an integer-valued random measure with compensator

$$\mu(dt\, dx; \omega) = m(t, dx; \omega)\, dt.$$

If furthermore

$$\int_0^1 x\, m(t, dx) < \infty, \tag{3.65}$$

then L_t may be expressed in the form

$$L_t = \int_0^t \int_0^1 x \, \Big(m(s, dx)\, ds + \tilde{M}(ds\, dx) \Big) \,,$$

where

$$\int_0^t \int_0^1 x \, \tilde{M}(ds\, dx),$$

is a $\mathbb{P}$-martingale. The point process $N_t = M([0,t] \times [0,1])$ represents the number of defaults and

$$\lambda_t(\omega) = \int_0^1 m(t, dx; \omega)$$

represents the default intensity. Denote by $T_1 \leq T_2 \leq ..$ the jump times of N. The cumulative loss process L may also be represented as

$$L_t = \sum_{k=1}^{N_t} Z_k,$$

where the "mark" Z_k, with values in $[0,1]$, is distributed according to

$$F_t(dx; \omega) = \frac{m_X(t, dx; \omega)}{\lambda_t(\omega)}.$$

Note that the percentage loss L_t belongs to $[0,1]$, so $\Delta L_t \in [0, 1 - L_{t-}]$. For the equity tranche $[0, K]$, we define the expected tranche notional at maturity T as

$$C_{t_0}(T, K) = \mathbb{E}[(K - L_T)_+ | \mathcal{F}_{t_0}]. \tag{3.66}$$

As noted in [28], the prices of portfolio credit derivatives such as CDO tranches only depend on the loss process through the expected tranche notionals. Therefore, if one is able to compute $C_{t_0}(T, K)$ then one is able to compute the values of all CDO tranches at date t_0. In the case of a loss process with constant loss increment, Cont and Savescu [29] derived a forward equation for the expected tranche notional. The following result generalizes the forward equation derived by Cont and Savescu [29] to a more general setting which allows for random, dependent loss sizes and possible dependence between the loss given default and the default intensity:

Proposition 3.5 (Forward equation for expected tranche notionals)**.** *Assume there exists a family $m_Y(t, dy, z)$ of measures on $[0,1]$ for all $t \in [t_0, T]$ and for all $A \in \mathcal{B}([0,1))$,*

$$m_Y(t, A, L_{t-}) = E[m_X(t, A, .)|L_{t-}], \tag{3.67}$$

Denote $M_Y(dt\, dy)$ the integer-valued random measure with compensator $m_Y(t, dy, z)\, dt$. Define the effective default intensity

$$\lambda^Y(t, z) = \int_0^{1-z} m_Y(t, dy, z). \tag{3.68}$$

Then the expected tranche notional $(T, K) \mapsto C_{t_0}(T, K)$, as a function of maturity and strike, is a solution of the partial integro-differential equation

$$\begin{aligned}
&\frac{\partial C_{t_0}}{\partial T}(T, K) \\
&\quad = -\int_0^K \frac{\partial^2 C_{t_0}}{\partial K^2}(T, dy) \left[\int_0^{K-y} (K - y - z)\, m_Y(T, dz, y) - (K - y)\lambda^Y(T, y)\right],
\end{aligned} \tag{3.69}$$

on $[t_0, \infty[\times]0, 1[$ with the initial condition: $\forall K \in [0,1]$,
$C_{t_0}(t_0, K) = (K - L_{t_0})_+$.

Proof. By replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_0}$ given $\mathcal{F}_0$, we may replace the conditional expectation in (3.66) by an expectation with respect to the marginal distribution $p_T(dy)$ of L_T under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. (3.66) can be expressed as

$$C(T, K) = \int_{\mathbb{R}^+} (K - y)^+\; p_T(dy). \tag{3.70}$$

Differentiating with respect to K, we get

$$\frac{\partial C}{\partial K} = \int_0^K p_T(dy) = \mathbb{E}\left[1_{\{L_{t-} \le K\}}\right], \qquad \frac{\partial^2 C}{\partial K^2}(T, dy) = p_T(dy). \tag{3.71}$$

For $h > 0$ applying the Tanaka-Meyer formula to $(K - L_t)^+$ between T and $T + h$, we have

$$\begin{aligned}
(K - L_{T+h})^+ &= (K - L_T)^+ - \int_T^{T+h} 1_{\{L_{t-} \le K\}} dL_t \\
&\quad + \sum_{T < t \le T+h} \left[(K - L_t)^+ - (K - L_{t-})^+ + 1_{\{L_{t-} \le K\}} \Delta L_t\right].
\end{aligned} \tag{3.72}$$

Taking expectations, we get

$$
\begin{aligned}
C(T+h,K)-C(T,K) &= \mathbb{E}\left[\int_T^{T+h} dt\, 1_{\{L_{t-}\le K\}} \int_0^{1-L_{t-}} x\, m(t,dx)\right] \\
&+ \mathbb{E}\left[\sum_{T<t\le T+h} (K-L_t)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}\Delta L_t\right].
\end{aligned}
$$

The first term may be computed as

$$
\begin{aligned}
& \mathbb{E}\left[\int_T^{T+h} dt\, 1_{\{L_{t-}\le K\}} \int_0^{1-L_{t-}} x\, m(t,dx)\right] \\
&= \int_T^{T+h} dt\, \mathbb{E}\left[1_{\{L_{t-}\le K\}} \int_0^{1-L_{t-}} x\, m(t,dx)\right] \\
&= \int_T^{T+h} dt\, \mathbb{E}\left[\mathbb{E}\left[1_{\{L_{t-}\le K\}} \int_0^{1-L_{t-}} x\, m(t,dx)\Big| L_{t-}\right]\right] \\
&= \int_T^{T+h} dt\, \mathbb{E}\left[1_{\{L_{t-}\le K\}} \int_0^{1-L_{t-}} x\, m_Y(t,dx,L_{t-})\right] \\
&= \int_T^{T+h} dt \int_0^K p_T(dy) \left(\int_0^{1-y} x\, m_Y(t,dx,y)\right).
\end{aligned}
$$

As for the jump term,

$$
\begin{aligned}
& \mathbb{E}\left[\sum_{T<t\le T+h} (K-L_t)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}\Delta L_t\right] \\
&= \mathbb{E}\left[\int_T^{T+h} dt \int_0^{1-L_{t-}} m(t,dx)\, \big((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}x\big)\right] \\
&= \int_T^{T+h} dt\, \mathbb{E}\left[\int_0^{1-L_{t-}} m(t,dx)\, \big((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}x\big)\right] \\
&= \int_T^{T+h} dt\, \mathbb{E}\left[\mathbb{E}\left[\int_0^{1-L_{t-}} m(t,dx)\, \big((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}x\big)\Big| L_{t-}\right]\right] \\
&= \int_T^{T+h} dt\, \mathbb{E}\left[\int_0^{1-L_{t-}} m_Y(t,dx,L_{t-})\, \big((K-L_{t-}-x)^+ - (K-L_{t-})^+ + 1_{\{L_{t-}\le K\}}x\big)\right] \\
&= \int_T^{T+h} dt \int_0^1 p_T(dy) \int_0^{1-y} m_Y(t,dx,y)\, \big((K-y-x)^+ - (K-y)^+ + 1_{\{y\le K\}}x\big),
\end{aligned}
$$

where the inner integrals may be computed as

$$
\begin{aligned}
& \int_0^1 p_T(dy) \int_0^{1-y} m_Y(t,dx,y) \left((K-y-x)^+ - (K-y)^+ + 1_{\{y\le K\}} x\right) \\
= & \int_0^K p_T(dy) \int_0^{1-y} m_Y(t,dx,y) \left((K-y-x) 1_{\{K-y>x\}} - (K-y-x)\right) \\
= & \int_0^K p_T(dy) \int_{K-y}^{1-y} m_Y(t,dx,y)(K-y-x).
\end{aligned}
$$

Gathering together all the terms, we obtain

$$
\begin{aligned}
& C(T+h,K) - C(T,K) \\
= & \int_T^{T+h} dt \int_0^K p_T(dy) \left(\int_0^{1-y} x\, m_Y(t,dx,y) \right) \\
+ & \int_T^{T+h} dt \int_0^K p_T(dy) \left(\int_{K-y}^{1-y} m_Y(t,dx,y)(K-y-x) \right) \\
= & \int_T^{T+h} dt \int_0^K p_T(dy) \left(-\int_0^{K-y} m_Y(t,dx,y)(K-y-x) + (K-y)\lambda^Y(T,y) \right).
\end{aligned}
$$

Dividing by h and taking the limit $h \to 0$ yields

$$
\begin{aligned}
\frac{\partial C}{\partial T} &= -\int_0^K p_T(dy) \left[\int_0^{K-y} (K-y-x)\, m_Y(T,dx,y) - (K-y)\lambda^Y(T,y) \right] \\
&= -\int_0^K \frac{\partial^2 C}{\partial K^2}(T,dy) \left[\int_0^{K-y} (K-y-x)\, m_Y(T,dx,y) - (K-y)\lambda^Y(T,y) \right].
\end{aligned}
$$

$\square$

In [29], loss given default (i.e. the jump size of L) is assumed constant $\delta = (1-R)/n$: then $Z_k = \delta$, so $L_t = \delta N_t$ and one can compute $C(T,K)$ using the law of N_t. Setting $t_0 = 0$ and assuming as above that $\mathcal{F}_{t_0}$ is generated by null sets, we have

$$
C(T,K) = \mathbb{E}[(K-L_T)^+] = \mathbb{E}[(k\,\delta - L_T)^+] = \delta\, \mathbb{E}[(k-N_T)^+] \equiv \delta\, C_k(T) \tag{3.73}
$$

The compensator of L_t is $\lambda_t\, \epsilon_\delta(dz)\, dt$, where $\epsilon_\delta(dz)$ is the point at the point δ. The effective compensator becomes

$$
m_Y(t,dz,y) = E[\lambda_t | L_{t-} = y]\, \epsilon_\delta(dz)\, dt = \lambda^Y(t,y)\, \epsilon_\delta(dz),
$$

and the effective default intensity is $\lambda^Y(t,y) = E[\lambda_t | L_{t-} = y]$. Using the notations in [29], if we set $y = j\delta$ then

$$\lambda^Y(t, j\delta) = E[\lambda_t | L_{t-} = j\delta] = E[\lambda_t | N_{t-} = j] = a_j(t)$$

and $p_t(dy) = \sum_{j=0}^{n} q_j(t)\epsilon_{j\delta}(dy)$. Let us focus on (3.69) in this case. We recall from the proof of Proposition 3.5 that

$$\begin{aligned}
\frac{\partial C}{\partial T}(T, k\delta) &= \int_0^1 p_T(dy) \int_0^{1-y} [(k\delta - y - z)^+ - (k\delta - y)^+]\, \lambda^Y(T, y)\, \epsilon_\delta(dz) \\
&= \int_0^1 p_T(dy)\, \lambda^Y(T, y)\, [(k\delta - y - \delta)^+ - (k\delta - y)^+]\, 1_{\{\delta < 1-y\}} \\
&= -\delta \sum_{j=0}^{n} q_j(T)\, a_j(T)\, 1_{\{j \le k-1\}}.
\end{aligned}$$

This expression can be simplified as in [29, Proposition 2], leading to the forward equation

$$\begin{aligned}
\frac{\partial C_k(T)}{\partial T} &= a_k(T)C_{k-1}(T) - a_{k-1}(T)C_k(T) - \sum_{j=1}^{k-2} C_j(T)[a_{j+1}(T) - 2a_j(T) + a_{j-1}(T)] \\
&= [a_k(T) - a_{k-1}(T)]C_{k-1}(T) - \sum_{j=1}^{k-2} (\nabla^2 a)_j C_j(T) - a_{k-1}(T)[C_k(T) - C_{k-1}(T)].
\end{aligned}$$

Hence we recover [29, Proposition 2] as a special case of Proposition 3.5.

Chapter 4

Short-time asymptotics for marginals of semimartingales

In this chapter, we study the short-time asymptotics of expectations of the form $E[f(\xi_t)]$ where ξ_t a discontinuous Itô semimartingale. We study two different cases: first, the case where f is a smooth (C^2) function, then the case where $f(x) = (x-K)_+$. We compute the leading term in the asymptotics in terms of the local characteristics of the semimartingale.

As an application, we derive the asymptotic behavior of call option prices close to maturity in a semimartingale model: whereas the behavior of *out-of-the-money* options is found to be linear in time, the short time asymptotics of *at-the-money* options is shown to depend on the fine structure of the semimartingale.

These results generalize and extend various asymptotic results previously derived for diffusion models [17, 18], Lévy processes [60, 44, 91], Markov jump-diffusion models [13] and one-dimensional martingales [78] to the more general case of a discontinuous semimartingale. In particular, the independence of increments or the Markov property do not play any role in our derivation.

4.1 Introduction

In applications such as stochastic control, statistics of processes and mathematical finance, one is often interested in computing or approximating conditional expectations of the type

$$\mathbb{E}\left[f(\xi_t)|\mathcal{F}_{t_0}\right], \tag{4.1}$$

where ξ is a stochastic process. Whereas for Markov process various well-known tools –partial differential equations, Monte Carlo simulation, semi-group methods– are available for the computation and approximation of conditional expectations, such tools do not carry over to the more general setting of semimartingales. Even in the Markov case, if the state space is high dimensional exact computations may be computationally prohibitive and there has been a lot of interest in obtaining approximations of (4.1) as $t \to t_0$. Knowledge of such *short-time asymptotics* is very useful not only for computation of conditional expectations but also for the estimation and calibration of such models. Accordingly, short-time asymptotics for (4.1) (which, in the Markov case, amounts to studying transition densities of the process ξ) has been previously studied for diffusion models [17, 18, 41], Lévy processes [60, 70, 83, 9, 44, 43, 91], Markov jump-diffusion models [3, 13] and one-dimensional martingales [78], using a variety of techniques. The proofs of these results in the case of Lévy processes makes heavy use of the independence of increments; proofs in other case rely on the Markov property, estimates for heat kernels for second-order differential operators or Malliavin calculus. What is striking, however, is the similarity of the results obtained in these different settings.

We reconsider here the short-time asymptotics of conditional expectations in a more general framework which contains existing models but allows to go beyond the Markovian setting and to incorporate path-dependent features. Such a framework is provided by the class of *Itô semimartingales*, which contains all the examples cited above but allows to use the tools of stochastic analysis. An *Itô semimartingale* on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ is a stochastic process ξ with the representation

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int_{\mathbb{R}^d} \kappa(y)\, \tilde{M}(ds\,dy) + \int_0^t \int_{\mathbb{R}^d} (y - \kappa(y))\; M(ds\,dy), \tag{4.2}$$

where ξ_0 is in $\mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, M is an integer-valued random measure on $[0, \infty] \times \mathbb{R}^d$ with compensator $\mu(\omega, dt, dy) = m(\omega, t, dy)dt$ and $\tilde{M} = M - \mu$ its compensated random measure, β (resp. δ) is an adapted process with values in $\mathbb{R}^d$ (resp. $M_{d\times n}(\mathbb{R})$) and

$$\kappa(y) = \frac{y}{1 + \|y\|^2}$$

is a truncation function.

We study the short-time asymptotics of conditional expectations of the form (4.1) where ξ is an Ito semimartingale of the form (4.2), for various classes of functions $f : \mathbb{R}^d \to \mathbb{R}$. First, we prove a general result for the case of $f \in C_b^2(\mathbb{R}^d, \mathbb{R})$. Then we will treat, when $d = 1$, the case of

$$\mathbb{E}\left[(\xi_t - K)^+ | \mathcal{F}_{t_0}\right], \tag{4.3}$$

which corresponds to the value at t_0 of a call option with strike K and maturity t in a model described by equation (4.2). We show that whereas the behavior of (4.3) in the case $K > \xi_{t_0}$ (*out-of-the-money* options) is linear in $t - t_0$, the asymptotics in the case $K = \xi_{t_0}$ (which corresponds to *at-the-money* options) depends on the fine structure of the semimartingale ξ at t_0. In particular, we show that for continuous semimartingales the short-maturity asymptotics of at-the-money options is determined by the local time of ξ at t_0. In each case we identify the leading term in the asymptotics and express this term in terms of the local characteristics of the semimartingale at t_0.

Our results unify various asymptotic results previously derived for particular examples of stochastic models and extend them to the more general case of a discontinuous semimartingale. In particular, we show that the independence of increments or the Markov property do not play any role in the derivation of such results.

Short-time asymptotics for expectations of the form (4.1) have been studied in the context of statistics of processes [60] and option pricing [3, 17, 18, 13, 44, 91, 78]. Berestycki, Busca and Florent [17, 18] derive short maturity asymptotics for call options when ξ_t is a diffusion, using analytical methods. Durrleman [38] studied the asymptotics of implied volatility in a general, non-Markovian stochastic volatility model. Jacod [60] derived asymptotics for (4.1) for various classes of functions f, when ξ_t is a Lévy process. Lopez [44] and Tankov [91] study the asymptotics of (4.3) when ξ_t is the exponential of a Lévy process. Lopez [44] also studies short-time asymptotic expansions for (4.1), by iterating the infinitesimal generator of the Lévy process ξ_t. Alos et al [3] derive short-maturity expansions for call options and implied volatility in a Heston model using Malliavin calculus. Benhamou et al. [13] derive short-maturity expansions for call options in a model where ξ is the solution of a Markovian SDE whose jumps are described by a compound Poisson process. These results apply to processes with independence of increments or solutions of a "Markovian" stochastic differential equation.

Durrleman studied the convergence of implied volatility to spot volatility in a stochastic volatility model with finite-variation jumps [37]. More recently, Nutz and Muhle-Karbe [78] study short-maturity asymptotics for call options in the case where ξ_t is a one-dimensional Itô semimartingale driven by a (one-dimensional) Poisson random measure whose Lévy measure is absolutely continuous. Their approach consists in "freezing" the characteristic triplet of ξ at t_0, approximating ξ_t by the corresponding Lévy process and using the results cited above [60, 44] to derive asymptotics for call option prices.

Our contribution is to extend these results to the more general case when ξ is a d-dimensional semimartingale with jumps. In contrast to previous derivations, our approach is purely based on Itô calculus, and makes no use of the Markov property or independence of increments. Also, our multidimensional setting allows to treat examples which are not accessible using previous results. For instance, when studying index options in jump-diffusion model (treated in the next chapter), one considers an index $I_t = \sum w_i S^i_t$ where $(S^1, ..., S^d)$ are Itô semimartingales. In this framework, I is indeed an Itô semimartingale whose stochastic integral representation is implied by those of S^i but it is naturally represented in terms of a d-dimensional integer-valued random measure, not a one-dimensional Poisson random measure. Our setting provides a natural framework for treating such examples.

Note that these 'short-time' asymptotics are different from the 'extreme-strike' asymptotics studied by Lee [71] and extended by Friz, Gulisashvili and others [12, 46, 49, 50]. But, in specific models, the two asymptotic regimes may be related using scaling arguments.

4.2 Short time asymptotics for conditional expectations

4.2.1 Main result

We make the following assumptions on the characteristics of the semimartingale ξ:

Assumption 4.1 (Right-continuity of characteristics at t_0).

$$\lim_{t\to t_0,\, t>t_0} \mathbb{E}\left[\|\beta_t - \beta_{t_0}\| | \mathcal{F}_{t_0}\right] = 0, \qquad \lim_{t\to t_0,\, t>t_0} \mathbb{E}\left[\|\delta_t - \delta_{t_0}\|^2 | \mathcal{F}_{t_0}\right] = 0,$$

where $\|.\|$ *denotes the Euclidean norm on* $\mathbb{R}^d$ *and for* $\varphi \in \mathcal{C}_0^b(\mathbb{R}^d \times \mathbb{R}^d, \mathbb{R})$,

$$\lim_{t\to t_0,\, t>t_0} \mathbb{E}\left[\int_{\mathbb{R}^d} \|y\|^2 \,\varphi(\xi_t, y)\, m(t, dy)|\mathcal{F}_{t_0}\right] = \int_{\mathbb{R}^d} \|y\|^2 \,\varphi(\xi_{t_0}, y)\, m(t_0, dy).$$

The second requirement, which may be viewed as a weak (right) continuity of $m(t, dy)$ along the paths of ξ, is satisfied for instance if $m(t, dy)$ is absolutely continuous with a density which is right-continuous in t at t_0.

Assumption 4.2 (Integrability condition)**.** $\exists T > t_0$,

$$\mathbb{E}\left[\int_{t_0}^{T} \|\beta_s\|\, ds\middle|\mathcal{F}_{t_0}\right] < \infty, \qquad \mathbb{E}\left[\int_{t_0}^{T} \|\delta_s\|^2\, ds\middle|\mathcal{F}_{t_0}\right] < \infty,$$
$$\mathbb{E}\left[\int_{t_0}^{T} \int_{\mathbb{R}^d} \|y\|^2\, m(s, dy)\, ds\middle|\mathcal{F}_{t_0}\right] < \infty.$$

Under these assumptions, the following result describes the asymptotic behavior of $\mathbb{E}\left[f(\xi_t)|\mathcal{F}_{t_0}\right]$ when $t \to t_0$:

Theorem 4.1. *Under Assumptions 4.1 and 4.2, for all* $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$,

$$\lim_{t \downarrow t_0} \frac{1}{t - t_0} \left(\mathbb{E}\left[f(\xi_t) | \mathcal{F}_{t_0} \right] - f(\xi_{t_0}) \right) = \mathcal{L}_{t_0} f(\xi_{t_0}). \tag{4.4}$$

where $\mathcal{L}_{t_0}$ *is the (random) integro-differential operator given by*

$$\begin{aligned} \forall f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R}), \quad \mathcal{L}_{t_0} f(x) &= \beta_{t_0}.\nabla f(x) + \frac{1}{2} \mathrm{tr}\left[{}^t\delta_{t_0} \delta_{t_0} \nabla^2 f \right](x) \qquad (4.5) \\ &+ \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1 + \|y\|^2}\, y.\nabla f(x)] m(t_0, dy). \end{aligned}$$

Before proving Theorem 4.1, we recall a useful lemma:

Lemma 4.1. *Let* $f : \mathbb{R}^+ \to \mathbb{R}$ *be right-continuous at* 0*, then*

$$\lim_{t \to 0} \frac{1}{t} \int_0^t f(s)\, ds = f(0). \tag{4.6}$$

Proof. Let F denote the primitive of f, then

$$\frac{1}{t} \int_0^t f(s)\, ds = \frac{1}{t} \left(F(t) - F(0) \right).$$

Letting $t \to 0^+$, this is nothing but the right derivative at 0 of F, which is $f(0)$ by right continuity of f. ☐

We can now prove Theorem 4.1.

Proof. of Theorem 4.1
We first note that, by replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_{t_0}}$ given $\mathcal{F}_{t_0}$, we may replace the conditional expectation in (4.4) by an expectation with respect to the marginal distribution of ξ_t under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, without loss of generality, we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. Let $f \in \mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$. Itô's formula

yields

$$
\begin{aligned}
f(\xi_t) &= f(\xi_0) + \int_0^t \nabla f(\xi_{s^-}) d\xi_s^i + \frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s \delta_s\right]\, ds \\
&+ \sum_{s\le t}\left[f(\xi_{s^-}+\Delta\xi_s) - f(\xi_{s^-}) - \sum_{i=1}^d \frac{\partial f}{\partial x_i}(\xi_{s^-})\Delta\xi_s^i\right] \\
&= f(\xi_0) + \int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds + \int_0^t \nabla f(\xi_{s^-}).\delta_s dW_s \\
&+ \frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s \delta_s\right]\ \ ds \\
&+ \int_0^t \int_{\mathbb{R}^d} \nabla f(\xi_{s^-}).\kappa(y)\, \tilde{M}(ds\, dy) \\
&+ \int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - \kappa(y).\nabla f(\xi_{s^-})\right)\, M(ds\, dy).
\end{aligned}
$$

We note that

- since ∇f is bounded and given Assumption 4.2, $\int_0^t \int_{\mathbb{R}^d} \nabla f(\xi_{s^-}).\kappa(y)\, \tilde{M}(ds\, dy)$ is a square-integrable martingale.
- since ∇f is bounded and given Assumption 4.2, $\int_0^t \nabla f(\xi_{s^-}).\delta_s dW_s$ is a martingale.

Hence, taking expectations, we obtain

$$
\begin{aligned}
\mathbb{E}\left[f(\xi_t)\right] &= \mathbb{E}\left[f(\xi_0)\right] + \mathbb{E}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] + \mathbb{E}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})^t\delta_s\delta_s\right]\, ds\right] \\
&+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - \kappa(y).\nabla f(\xi_{s^-})\right)\, M(ds\, dy)\right] \\
&= \mathbb{E}\left[f(\xi_0)\right] + \mathbb{E}\left[\int_0^t \nabla f(\xi_{s^-}).\beta_s\, ds\right] + \mathbb{E}\left[\frac{1}{2}\int_0^t \mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\, {}^t\delta_s\delta_s\right]\, ds\right] \\
&+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \left(f(\xi_{s^-}+y) - f(\xi_{s^-}) - \kappa(y).\nabla f(\xi_{s^-})\right)\, m(s, dy)\, ds\right],
\end{aligned}
$$

that is

$$
\mathbb{E}\left[f(\xi_t)\right] = \mathbb{E}\left[f(\xi_0)\right] + \mathbb{E}\left[\int_0^t \mathcal{L}_s f(\xi_s)\, ds\right]. \tag{4.7}
$$

where $\mathcal{L}$ denote the integro-differential operator given, for all $t \in [0,T]$ and for all $f \in \mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$, by

$$\begin{aligned}\mathcal{L}_t f(x) &= \beta_t.\nabla f(x) + \frac{1}{2}\mathrm{tr}\left[{}^t\delta_t\delta_t\nabla^2 f\right](x) \\ &\quad + \int_{\mathbb{R}^d}[f(x+y) - f(x) - \frac{1}{1+\|y\|^2}\, y.\nabla f(x)]m(t,dy),\end{aligned} \tag{4.8}$$

Equation (4.7) yields

$$\begin{aligned}&\frac{1}{t}\,\mathbb{E}\left[f(\xi_t)\right] - \frac{1}{t}f(\xi_0) - \mathcal{L}_0 f(\xi_0)\\ =\;& \mathbb{E}\left[\frac{1}{t}\int_0^t ds\;(\nabla f(\xi_s).\beta_s - \nabla f(\xi_0).\beta_0)\right]\\ +\;& \frac{1}{2}\,\mathbb{E}\left[\frac{1}{t}\int_0^t ds\,\mathrm{tr}\left[\nabla^2 f(\xi_s)\,{}^t\delta_s\delta_s - \nabla^2 f(\xi_0)\,{}^t\delta_0\delta_0\right]\right]\\ +\;& \mathbb{E}\Big[\int_{\mathbb{R}^d}\frac{1}{t}\int_0^t\; ds\,\big[m(s,dy)\;(f(\xi_s+y) - f(\xi_s) - \kappa(y).\nabla f(\xi_s))\\ &\qquad\qquad -m(0,dy),(f(\xi_0+y) - f(\xi_0) - \kappa(y).\nabla f(\xi_0))\,\big]\Big].\end{aligned}$$

Define

$$\begin{aligned}\Delta_1(t) &= \mathbb{E}\left[\frac{1}{t}\int_0^t ds\;(\nabla f(\xi_s).\beta_s - \nabla f(\xi_0).\beta_0)\right],\\ \Delta_2(t) &= \frac{1}{2}\,\mathbb{E}\left[\frac{1}{t}\int_0^t ds\,\mathrm{tr}\left[\nabla^2 f(\xi_{s^-})\,{}^t\delta_s\delta_s - \nabla^2 f(\xi_0)\,{}^t\delta_0\delta_0\right]\right],\\ \Delta_3(t) &= \mathbb{E}\Big[\int_{\mathbb{R}^d}\frac{1}{t}\int_0^t\; ds\,\big[m(s,dy)\;(f(\xi_s+y) - f(\xi_s) - \kappa(y).\nabla f(\xi_{s^-}))\\ &\qquad\qquad -m(0,dy)\;(f(\xi_0+y) - f(\xi_0) - \kappa(y).\nabla.f(\xi_0))\,\big]\Big].\end{aligned}$$

Thanks to Assumptions 4.1 and 4.2,

$$\mathbb{E}\left[\int_0^t ds\;|\nabla f(\xi_s).\beta_s - \nabla f(\xi_0).\beta_0|\right] \le \mathbb{E}\left[\int_0^t ds\,\|\nabla f\|\,(\|\beta_s\| + \|\beta_0\|)\right] < \infty.$$

Fubini's theorem then applies:

$$\Delta_1(t) = \frac{1}{t}\int_0^t ds\,\mathbb{E}\left[\nabla f(\xi_s).\beta_s - \nabla f(\xi_0).\beta_0\right].$$

Let us prove that

$$
\begin{aligned}
g_1 : [0, T[&\to \mathbb{R} \\
t &\to \mathbb{E}\left[\nabla f(\xi_t).\beta_t - \nabla f(\xi_0).\beta_0\right],
\end{aligned}
$$

is right-continuous at 0 with $g_1(0) = 0$, yielding $\Delta_1(t) \to 0$ when $t \to 0^+$ if one applies Lemma 4.1.

$$
\begin{aligned}
|g_1(t)| &= \left|\mathbb{E}\left[\nabla f(\xi_t).\beta_t - \nabla f(\xi_0).\beta_0\right]\right| \\
&= \left|\mathbb{E}\left[\left(\nabla f(\xi_t) - \nabla f(\xi_0)\right).\beta_0 + \nabla f(\xi_t).\left(\beta_t - \beta_0\right)\right]\right| \\
&\leq \|\nabla f\|_\infty \mathbb{E}\left[\|\beta_t - \beta_0\|\right] + \|\beta_0\| \, \|\nabla^2 f\|_\infty \mathbb{E}\left[\|\xi_t - \xi_0\|\right],
\end{aligned} \tag{4.9}
$$

where $\|\|_\infty$ denotes the supremum norm on $\mathcal{C}_b^2(\mathbb{R}^d, \mathbb{R})$. Assumption 4.1 implies that:

$$
\lim_{t \to 0^+} \mathbb{E}\left[\|\beta_t - \beta_0\|\right] = 0.
$$

Thanks to Assumption 4.2, one may decompose ξ_t as follows

$$
\begin{aligned}
\xi_t &= \xi_0 + A_t + M_t, \\
A_t &= \int_0^t \left(\beta_s \, ds + \int_{\mathbb{R}^d} (y - \kappa(y)) \, m(s, dy)\right) ds, \\
M_t &= \int_0^t \delta_s \, dW_s + \int_0^t \int_{\mathbb{R}^d} y \, \tilde{M}(ds\, dy),
\end{aligned} \tag{4.10}
$$

where A_t is of finite variation and M_t is a local martingale. First, applying Fubini's theorem (using Assumption 4.2),

$$
\begin{aligned}
\mathbb{E}\left[\|A_t\|\right] \;&\leq\; \mathbb{E}\left[\int_0^t \|\beta_s\| \, ds\right] + \mathbb{E}\left[\int_0^t \int_{\mathbb{R}^d} \|y - \kappa(y)\| \, m(s, dy) \, ds\right] \\
&=\; \int_0^t ds \, \mathbb{E}\left[\|\beta_s\|\right] + \int_0^t ds \, \mathbb{E}\left[\int_{\mathbb{R}^d} \|y - \kappa(y)\| \, m(s, dy)\right].
\end{aligned}
$$

Thanks to Assumption 4.1, one observes that if $s \in [0, T[\to \mathbb{E}\left[\|\beta_s - \beta_0\|\right]$ is right-continuous at 0 so is $s \in [0, T[\to \mathbb{E}\left[\|\beta_s\|\right]$. Furthermore, Assumption 4.1 yields that

$$
s \in [0, T[\to \mathbb{E}\left[\int_{\mathbb{R}^d} \|y - \kappa(y)\| \, m(s, dy)\right]
$$

is right-continuous at 0 and Lemma 4.1 implies that

$$\lim_{t \to 0^+} \mathbb{E}\left[\|A_t\|\right] = 0.$$

Furthermore, writing $M_t = (M_t^1, \cdots, M_t^d)$,

$$\mathbb{E}\left[\|M_t\|^2\right] = \sum_{1 \le i \le d} \mathbb{E}\left[|M_t^i|^2\right].$$

Burkholder's inequality [81, Theorem IV.73] implies that there exists $C > 0$ such that

$$\begin{aligned}
\sup_{s \in [0,t]} \mathbb{E}\left[|M_s^i|^2\right] &\le C\,\mathbb{E}\left[[M^i, M^i]_t\right] \\
&= C\,\mathbb{E}\left[\int_0^t ds\, |\delta_s^i|^2 + \int_0^t ds \int_{\mathbb{R}^d} |y_i|^2 m(s, dy)\right].
\end{aligned}$$

Using Assumption 4.2 we may apply Fubini's theorem to obtain

$$\begin{aligned}
\sup_{s \in [0,t]} \mathbb{E}\left[\|M_s\|^2\right] &\le C \sum_{1 \le i \le d} \mathbb{E}\left[\int_0^t ds\, |\delta_s^i|^2\right] + \mathbb{E}\left[\int_0^t ds \int_{\mathbb{R}^d} |y_i|^2 m(s, dy)\right] \\
&= C\left(\mathbb{E}\left[\int_0^t ds\, \|\delta_s\|^2\right] + \mathbb{E}\left[\int_0^t ds \int_{\mathbb{R}^d} \|y\|^2 m(s, dy)\right]\right) \\
&= C\left(\int_0^t ds\, \mathbb{E}\left[\|\delta_s\|^2\right] + \int_0^t ds\, \mathbb{E}\left[\int_{\mathbb{R}^d} \|y\|^2 m(s, dy)\right]\right).
\end{aligned}$$

Thanks to Assumption 4.1, Lemma 4.1 yields

$$\lim_{t \to 0^+} \mathbb{E}\left[\|M_t\|^2\right] = 0.$$

Using the Jensen inequality, one obtains

$$\mathbb{E}\left[\|M_t\|\right] = \mathbb{E}\left[\sqrt{\sum_{1 \le i \le d} |M_t^i|^2}\right] \le \sqrt{\mathbb{E}\left[\sum_{1 \le i \le d} |M_t^i|^2\right]} = \mathbb{E}\left[\|M_t\|^2\right].$$

Hence,

$$\lim_{t \to 0^+} \mathbb{E}\left[\|M_t\|\right] = 0,$$

and

$$\lim_{t\to 0^+} \mathbb{E}\left[\|\xi_t-\xi_0\|\right] \le \lim_{t\to 0^+} \mathbb{E}\left[\|A_t\|\right] + \lim_{t\to 0^+} \mathbb{E}\left[\|M_t\|\right] = 0.$$

Going back to the inequalities (4.9), one obtains

$$\lim_{t\to 0^+} g_1(t) = 0.$$

Similarly, $\Delta_2(t) \to 0$ and $\Delta_3(t) \to 0$ when $t \to 0^+$. This ends the proof. $\square$

Remark 4.1. *In applications where a process is constructed as the solution to a stochastic differential equation driven by a Brownian motion and a Poisson random measure, one usually starts from a representation of the form*

$$\zeta_t = \zeta_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dW_s + \int_0^t \int \psi_s(y)\, \tilde{N}(ds\, dy), \tag{4.11}$$

where $\xi_0 \in \mathbb{R}^d$, W is a standard $\mathbb{R}^n$-valued Wiener process, β and δ are non-anticipative càdlàg processes, N is a Poisson random measure on $[0,T]\times\mathbb{R}^d$ with intensity $\nu(dy)\, dt$ where ν is a Lévy measure

$$\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu(dy) < \infty, \qquad \tilde{N} = N - \nu(dy)dt,$$

and $\psi : [0,T]\times\Omega\times\mathbb{R}^d \mapsto \mathbb{R}^d$ is a predictable random function representing jump amplitude. This representation is different from (4.2)*, but Lemma 2.2 in Chapter 2 shows that one can switch from the representation (4.11) to the representation (4.2) in an explicit manner.*

In particular, if one rewrites Assumption 4.1 in the framework of equation (4.11), one recovers the Assumptions of [78] as a special case.

4.2.2 Some consequences and examples

If we have further information on the behavior of f in the neighborhood of ξ_0, then the quantity $L_0 f(\xi_0)$ ca be computed more explicitly. We summarize some commonly encountered situations in the following Proposition.

Proposition 4.1. *Under Assumptions 4.1 and 4.2,*

1. *If $f(\xi_0) = 0$ and $\nabla f(\xi_0) = 0$, then*

$$\lim_{t\to 0^+} \frac{1}{t}\, \mathbb{E}\left[f(\xi_t)\right] = \frac{1}{2}\mathrm{tr}\left[{}^t\delta_0\delta_0\, \nabla^2 f(\xi_0)\right] + \int_{\mathbb{R}^d} f(\xi_0+y)\, m(0,dy). \tag{4.12}$$

2. If furthermore $\nabla^2 f(\xi_0) = 0$, *then*

$$\lim_{t\to 0^+} \frac{1}{t}\,\mathbb{E}\left[f(\xi_t)\right] = \int_{\mathbb{R}^d} f(\xi_0 + y)\, m(0, dy). \tag{4.13}$$

Proof. Applying Theorem 4.1, $\mathcal{L}_0 f(\xi_0)$ writes

$$\begin{aligned}\mathcal{L}_0 f(\xi_0) &= \beta_0 . \nabla f(\xi_0) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(\xi_0)\, {}^t\delta_0\delta_0\right](\xi_0)\\ &\quad + \int_{\mathbb{R}^d} [f(\xi_0 + y) - f(\xi_0) - \frac{1}{1+\|y\|^2}\, y.\nabla f(\xi_0)] m(0, dy).\end{aligned}$$

The proposition follows immediately. □

Remark 4.2. *As observed by Jacod [60, Section 5.8] in the setting of Lévy processes, if* $f(\xi_0) = 0$ *and* $\nabla f(\xi_0) = 0$, *then* $f(x) = O(\|x - \xi_0\|^2)$. *If furthermore* $\nabla^2 f(\xi_0) = 0$, *then* $f(x) = o(\|x - \xi_0\|^2)$.

Let us now compute in a more explicit manner the asymptotics of (4.1) for specific semimartingales.

Functions of a Markov process

An important situations which often arises in applications is when a stochastic processe ξ is driven by an underlying Markov process, i.e.

$$\xi_t = f(Z_t) \qquad f \in C^2(\mathbb{R}^d, \mathbb{R}), \tag{4.14}$$

where Z_t is a Markov process, defined as the weak solution on $[0, T]$ of a stochastic differential equation

$$\begin{aligned} Z_t = Z_0 &+ \int_0^t b(u, Z_{u-})\, du + \int_0^t \Sigma(u, Z_{u-})\, dW_u \\ &+ \int_0^t \int \psi(u, Z_{u-}, y)\, \tilde{N}(du\ dy), \end{aligned} \tag{4.15}$$

where (W_t) is an n-dimensional Brownian motion, N is a Poisson random measure on $[0, T] \times \mathbb{R}^d$ with Lévy measure $\nu(y)\, dy$, $\tilde{N}$ the associated compensated random measure, $\Sigma : [0, T] \times \mathbb{R}^d \mapsto M_{d\times d}(\mathbb{R})$, $b : [0, T] \times \mathbb{R}^d \mapsto \mathbb{R}^d$

and $\psi : [0,T] \times \mathbb{R}^d \times \mathbb{R}^d$ are measurable functions such that

$$\psi(.,.,0) = 0 \qquad \psi(t,z,.) \text{ is a } \mathcal{C}^1(\mathbb{R}^d,\mathbb{R}^d) - \text{diffeomorphism}$$
$$\forall t \in [0,T], \quad \mathbb{E}\left[\int_0^t \int_{\{\|y\|\geq 1\}} \sup_{z\in\mathbb{R}^d} \left(1 \wedge \|\psi(s,z,y)\|^2\right) \nu(y)\, dy\, ds\right] < \infty. \tag{4.16}$$

In this setting, as shown in Proposition 2.3, one may verify the regularity of the Assumptions 4.1 and Assumption 4.2 by requiring mild and easy-to-check assumptions on the coefficients:

Assumption 4.3. *$b(.,.)$, $\Sigma(.,.)$ and $\psi(.,.,y)$ are continuous in the neighborhood of $(0,Z_0)$*

Assumption 4.4. *There exist $T>0, R>0$ such that*

$$\text{Either} \qquad \forall t \in [0,T] \qquad \inf_{\|z-Z_0\|\leq R} \ \inf_{x\in\mathbb{R}^d,\, \|x\|=1} {}^t x.\Sigma(t,z).x > 0$$
$$\text{or} \qquad \Sigma \equiv 0.$$

We then obtain the following result:

Proposition 4.2. *Let $f \in \mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$ such that*

$$\forall (z_1,\cdots,z_{d-1}) \in \mathbb{R}^{d-1}, \quad u \mapsto f(z_1,\ldots,z_{d-1},u) \text{ is a } \mathcal{C}^1(\mathbb{R},\mathbb{R}) - \text{diffeomorphism}.$$

Define

$$\begin{cases} \beta_0 & = \nabla f(Z_0).b(0,Z_0) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_0)\, {}^t\Sigma(0,Z_0)\Sigma(0,Z_0)\right] \\ & \quad + \int_{\mathbb{R}^d} \left(f(Z_0+\psi(0,Z_0,y)) - f(Z_0) - \psi(0,Z_0,y).\nabla f(Z_0)\right) \nu(y)\, dy, \\ \delta_0 & = \|\nabla f(Z_0)\Sigma(0,Z_0)\|, \end{cases}$$

and the measure $m(0,.)$ via

$$\begin{aligned} m(0,[u,\infty[) &= \int_{\mathbb{R}^d} 1_{\{f(Z_0+\psi(0,Z_0,y))-f(Z_0)\geq u\}}\, \nu(y)\, dy \quad u>0, \\ m(0,[-\infty,u]) &= \int_{\mathbb{R}^d} 1_{\{f(Z_0+\psi(0,Z_0,y))-f(Z_0)\leq u\}}\, \nu(y)\, dy \quad u<0. \end{aligned} \tag{4.17}$$

Under the Assumptions 4.3 and 4.4, $\forall g \in \mathcal{C}_b^2(\mathbb{R}^d,\mathbb{R})$,

$$\lim_{t\to 0^+} \frac{\mathbb{E}\left[g(\xi_t)\right] - g(\xi_0)}{t} = \beta_0\, g'(\xi_0) + \frac{\delta_0^2}{2}\, g''(\xi_0) + \int_{\mathbb{R}^d} \left[g(\xi_0+u) - g(\xi_0) - u g'(\xi_0)\right] m(0,du). \tag{4.18}$$

Proof. Under the conditions (4.16) and the Assumption 4.4, Proposition 2.3 shows that ξ_t admits the semimartingale decomposition

$$\xi_t = \xi_0 + \int_0^t \beta_s\, ds + \int_0^t \delta_s\, dB_s + \int_0^t \int u\, \tilde{K}(ds\, du),$$

where

$$\begin{cases} \beta_t & = \nabla f(Z_{t^-}).b(t, Z_{t-}) + \frac{1}{2}\mathrm{tr}\left[\nabla^2 f(Z_{t-})^t \Sigma(t, Z_{t-})\Sigma(t, Z_{t-})\right] \\ & \quad + \int_{\mathbb{R}^d} \left(f(Z_{t^-} + \psi(t, Z_{t-}, y)) - f(Z_{t^-}) - \psi(t, Z_{t-}, y).\nabla f(Z_{t^-})\right) \nu(y)\, dy, \\ \delta_t & = \|\nabla f(Z_{t-})\Sigma(t, Z_{t-})\|, \end{cases}$$

and K is an integer-valued random measure on $[0, T] \times \mathbb{R}$ with compensator $k(t, Z_{t-}, u)\, du\, dt$ defined via

$$\begin{aligned} k(t, Z_{t-}, u) \quad = \quad & \int_{\mathbb{R}^{d-1}} |\det \nabla_y \Phi(t, Z_{t-}, (y_1, \cdots, y_{d-1}, u))| \\ & \qquad \nu(\Phi(t, Z_{t-}, (y_1, \cdots, y_{d-1}, u)))\, dy_1 \cdots dy_{d-1}, \end{aligned}$$

with

$$\begin{cases} \Phi(t, z, y) = \phi(t, z, \kappa_z^{-1}(y)) \quad \kappa_z^{-1}(y) = (y_1, \cdots, y_{d-1}, F_z(y)), \\ F_z(y) : \mathbb{R}^d \to \mathbb{R} \quad f(z + (y_1, \cdots, y_{d-1}, F_z(y))) - f(z) = y_d. \end{cases}$$

From Assumption 4.3 it follows that Assumptions 4.1 and 4.2 hold for β_t, δ_t and $k(t, Z_t-, .)$ on $[0, T]$. Applying Theorem 4.1, the result follows immediately. □

Remark 4.3. *Benhamou et al. [13] studied the case where Z_t is the solution of a 'Markovian' SDE whose jumps are given by a compound Poisson Process. The above results generalizes their result to the (general) case where the jumps are driven by an arbitrary integer-valued random measure.*

Time-changed Lévy processes

Models based on time–changed Lévy processes provide another class of examples of non-Markovian models which have generated recent interest in mathematical finance. Let L_t be a real-valued Lévy process, (b, σ^2, ν) be its characteristic triplet, N its jump measure. Define

$$\xi_t = L_{\Theta_t} \qquad \Theta_t = \int_0^t \theta_s ds, \tag{4.19}$$

where (θ_t) is a locally bounded $\mathcal{F}_t$-adapted positive càdlàg process, interpreted as the rate of time change.

Proposition 4.3. *If*

$$\int_{\mathbb{R}} |y|^2 \, \nu(dy) < \infty \quad \text{and} \quad \lim_{t \to 0,\, t>0} \mathbb{E}\left[|\theta_t - \theta_0|\right] = 0 \tag{4.20}$$

then

$$\forall f \in \mathcal{C}_b^2(\mathbb{R}, \mathbb{R}), \qquad \lim_{t \to 0^+} \frac{\mathbb{E}\left[f(\xi_t)\right] - f(\xi_0)}{t} = \theta_0 \, \mathcal{L}_0 f(\xi_0) \tag{4.21}$$

where $\mathcal{L}_0$ is the infinitesimal generator of the L:

$$\mathcal{L}_0 f(x) = b\, f'(x) + \frac{\sigma^2}{2} f''(x) + \int_{\mathbb{R}^d} [f(x+y) - f(x) - \frac{1}{1+|y|^2} \, y f'(x)] \nu(dy). \tag{4.22}$$

Proof. Considering the Lévy-Itô decomposition of L:

$$\begin{aligned} L_t &= \left(b - \int_{\{|y| \le 1\}} (y - \kappa(y)) \; \nu(dy)\right) t + \sigma W_t \\ &+ \int_0^t \int_{\mathbb{R}} \kappa(z) \tilde{N}(dsdz) + \int_0^t \int_{\mathbb{R}} (z - \kappa(z))\, N(dsdz), \end{aligned}$$

then, as shown in the proof of Theorem 2.6, ξ has the representation

$$\begin{aligned} \xi_t = \xi_0 + \int_0^t \sigma \sqrt{\theta_s}\, dZ_s + \int_0^t \left(b - \int_{\{|y| \le 1\}} (y - \kappa(y)) \; \nu(dy)\right) \theta_s \, ds \\ + \int_0^t \int_{\mathbb{R}} \kappa(z) \theta_s \, \tilde{N}(ds\, dz) + \int_0^t \int_{\mathbb{R}} (z - \kappa(z))\, \theta_s \, N(ds\, dz). \end{aligned}$$

where Z is a Brownian motion. With the notation of equation (4.2), one identifies

$$\beta_t = \left(b - \int_{\{|y| \le 1\}} (y - \kappa(y)) \; \nu(dy)\right) \theta_t, \quad \delta_t = \sigma \sqrt{\theta_t}, \quad m(t, dy) = \theta_t \, \nu(dy).$$

If (4.20) holds, then Assumptions 4.1 and 4.2 hold for (β, δ, m) and Theorem 4.1 may be applied to obtain the result. $\square$

4.3 Short-maturity asymptotics for call options

Consider a (strictly positive) price process S whose dynamics under the pricing measure $\mathbb{P}$ is given by a stochastic volatility model with jumps:

$$S_t = S_0 + \int_0^t r(s) S_{s^-} ds + \int_0^t S_{s^-} \delta_s dW_s + \int_0^t \int_{-\infty}^{+\infty} S_{s^-}(e^y - 1)\tilde{M}(ds\ dy), \tag{4.23}$$

where $r(t) > 0$ represents a (deterministic) bounded discount rate. For convenience, we shall assume that $r \in \mathcal{C}_0^b(\mathbb{R}^+, \mathbb{R}^+)$. δ_t represents the volatility process and M is an integer-valued random measure with compensator $\mu(\omega; dt\, dy) = m(\omega; t, dy)\, dt$, representing jumps in the log-price, and $\tilde{M} = M - \mu$ its compensated random measure. We make the following assumptions on the characteristics of S:

Assumption 4.5 (Right-continuity at t_0)**.**

$$\lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[|\delta_t - \delta_{t_0}|^2 | \mathcal{F}_{t_0}\right] = 0.$$

For all $\varphi \in \mathcal{C}_0^b(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$,

$$\lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[\int_{\mathbb{R}} \left(e^{2y} \wedge |y|^2\right)\ \varphi(S_t, y)\, m(t, dy) | \mathcal{F}_{t_0}\right] = \int_{\mathbb{R}} \left(e^{2y} \wedge |y|^2\right)\ \varphi(S_{t_0}, y)\, m(t_0, dy).$$

Assumption 4.6 (Integrability condition)**.**

$$\exists T > t_0, \quad \mathbb{E}\left[\exp\left(\frac{1}{2}\int_{t_0}^T \delta_s^2\, ds + \int_{t_0}^T ds \int_{\mathbb{R}} (e^y - 1)^2 m(s, dy)\right) | \mathcal{F}_{t_0}\right] < \infty \quad .$$

We recall that the value $C_{t_0}(t, K)$ at time t_0 of a call option with expiry $t > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(t, K) = e^{-\int_{t_0}^t r(s)\, ds} \mathbb{E}[\max(S_t - K, 0) | \mathcal{F}_{t_0}]. \tag{4.24}$$

The discounted asset price

$$\hat{S}_t = e^{-\int_{t_0}^t r(u)\, du}\, S_t,$$

is the stochastic exponential of the martingale ξ defined by

$$\xi_t = \int_0^t \delta_s \, dW_s + \int_0^t \int (e^y - 1) \tilde{M}(ds\, dy).$$

Under Assumption 4.6, we have

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\langle \xi, \xi \rangle_T^d + \langle \xi, \xi \rangle_T^c\right)\right] < \infty,$$

where $\langle \xi, \xi \rangle^c$ and $\langle \xi, \xi \rangle^d$ denote the continuous and purely discontinuous parts of $[\xi, \xi]$ and [80, Theorem 9] implies that $(\hat{S}_t)_{t \in [t_0, T]}$ is a $\mathbb{P}$-martingale. In particular the expectation in (4.24) is finite.

4.3.1 Out-of-the money call options

We first study the asymptotics of out-of-the money call options i.e. the case where $K > S_{t_0}$. The main result is as follows:

Theorem 4.2 (Short-maturity behavior of out-of-the money options)**.** *Under Assumption 4.5 and Assumption 4.6, if $S_{t_0} < K$ then*

$$\frac{1}{t - t_0} C_{t_0}(t, K) \underset{t \to t_0^+}{\longrightarrow} \int_0^\infty (S_{t_0} e^y - K)_+ m(t_0, dy). \tag{4.25}$$

This limit can also be expressed using the exponential double tail ψ_{t_0} of the compensator, defined as

$$\psi_{t_0}(z) = \int_z^{+\infty} dx \; e^x \int_x^\infty m(t_0, du) \quad z > 0. \tag{4.26}$$

Then, as shown in [14, Lemma 1],

$$\int_0^\infty (S_{t_0} e^y - K)_+ m(t_0, dy) = S_{t_0} \psi_{t_0}\left(\ln\left(\frac{K}{S_{t_0}}\right)\right).$$

Proof. The idea is to apply Theorem 4.1 to smooth approximations f_n of the function $x \to (x - K)^+$ and conclude using a dominated convergence argument.

First, as argued in the proof of Theorem 4.1, we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. Applying the Itô formula to $X_t \equiv \ln(S_t)$, we obtain

$$
\begin{aligned}
X_t &= \ln(S_0) + \int_0^t \frac{1}{S_{s^-}}\, dS_s + \frac{1}{2}\int_0^t \frac{-1}{S_{s^-}^2}\,(S_{s-}\delta_s)^2\, ds \\
&+ \sum_{s\le t}\left[\ln(S_{s^-}+\Delta S_s) - \ln(S_{s^-}) - \frac{1}{S_{s^-}}\Delta S_s\right] \\
&= \ln(S_0) + \int_0^t \left(r(s) - \frac{1}{2}\delta_s^2\right)\, ds + \int_0^t \delta_s\, dW_s \\
&+ \int_0^t \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} (e^y - 1 - y)\ M(ds\,dy)
\end{aligned}
$$

Note that there exists $C > 0$ such that

$$
|e^y - 1 - y\,\frac{1}{1+|y|^2}| \le C\ (e^y-1)^2\,.
$$

Thanks to Jensen's inequality, Assumption 4.6 implies that this quantity is finite, allowing us to write

$$
\begin{aligned}
& \int_0^t \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} (e^y - 1 - y)\ M(ds\,dy) \\
=\ & \int_0^t \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,\frac{1}{1+|y|^2}\right)\ M(ds\,dy) \\
+\ & \int_0^t \int_{-\infty}^{+\infty} \left(y - y\frac{1}{1+|y|^2}\right)\ M(ds\,dy) \\
=\ & \int_0^t \int_{-\infty}^{+\infty} (e^y - 1)\tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,\frac{1}{1+|y|^2}\right)\ \tilde{M}(ds\,dy) \\
-\ & \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,\frac{1}{1+|y|^2}\right)\ m(s,y)\,ds\,dy + \int_0^t \int_{-\infty}^{+\infty} \left(y - y\frac{1}{1+|y|^2}\right)\ M(ds\,dy) \\
=\ & \int_0^t \int_{-\infty}^{+\infty} y\,\frac{1}{1+|y|^2}\ \tilde{M}(ds\,dy) - \int_0^t \int_{-\infty}^{+\infty} \left(e^y - 1 - y\,\frac{1}{1+|y|^2}\right)\ m(s,y)\,ds\,dy \\
+\ & \int_0^t \int_{-\infty}^{+\infty} \left(y - y\frac{1}{1+|y|^2}\right)\ M(ds\,dy).
\end{aligned}
$$

We can thus represent X_t as in (4.2)):

$$X_t = X_0 + \int_0^t \beta_s\, dt + \int_0^t \delta_s\, dW_s \\ + \int_0^t \int_{-\infty}^{+\infty} y\, \frac{1}{1+|y|^2}\, \tilde{M}(ds\, dy) + \int_0^t \int_{-\infty}^{+\infty} \left(y - y\, \frac{1}{1+|y|^2} \right) M(ds\, dy), \tag{4.27}$$

with

$$\beta_t = r(t) - \frac{1}{2}\delta_t^2 - \int_{-\infty}^{\infty} \left(e^y - 1 - y\, \frac{1}{1+|y|^2} \right) m(t,y)\, dt\, dy.$$

Hence, if δ and $m(.,dy)$ satisfy Assumption 4.5 then β, δ and $m(.,dy)$ satisfy Assumption 4.1. Thanks to Jensen's inequality, Assumption 4.6 implies that β, δ and m satisfy Assumption 4.2. One may apply Theorem 4.1 to X_t for any function of the form $f \circ exp$, $f \in \mathcal{C}_b^2(\mathbb{R},\mathbb{R})$. Let us introduce a family $f_n \in \mathcal{C}_b^2(\mathbb{R},\mathbb{R})$ such that

$$\begin{cases} f_n(x) = (x-K)^+ & |x-K| > \frac{1}{n} \\ (x-K)^+ \le f_n(x) \le \frac{1}{n} & |x-K| \le \frac{1}{n}. \end{cases}$$

Then for $x \neq K$, $f_n(x) \underset{n\to\infty}{\longrightarrow} (x-K)^+$. Define, for $f \in C_0^\infty(\mathbb{R}^+,\mathbb{R})$,

$$\mathcal{L}_0 f(x) = r(0) x f'(x) + \frac{x^2 \delta_0^2}{2} f''(x) \\ + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y - 1).f'(x)] m(0,dy). \tag{4.28}$$

First, observe that if $N_1 \ge 1/|S_0 - K|$,

$$\forall n \ge N_1, f_n(S_0) = (S_0 - K)^+ = 0, \quad \text{so}$$

$$\frac{1}{t}\mathbb{E}\left[(S_t - K)^+\right] \le \frac{1}{t}\mathbb{E}\left[f_n(S_t)\right] = \frac{1}{t}\left(\mathbb{E}\left[f_n(S_t)\right] - f_n(S_0)\right).$$

Letting $t \to 0^+$ yields

$$\limsup_{t\to 0^+} \frac{1}{t}\, e^{-\int_0^t r(s)\, ds}\, \mathbb{E}\left[(S_t - K)^+\right] \le \mathcal{L}_0 f_n(S_0). \tag{4.29}$$

Furthermore,

$$\begin{aligned}
\mathbb{E}\left[(S_t-K)^+\right] &\geq \mathbb{E}\left[f_n(S_t)1_{\{|S_t-K|>\frac{1}{n}\}}\right] \\
&= \mathbb{E}\left[f_n(S_t)\right]-\mathbb{E}\left[f_n(S_t)1_{\{|S_t-K|\leq\frac{1}{n}\}}\right] \\
&\geq \mathbb{E}\left[f_n(S_t)\right]-f_n(S_0)-\frac{1}{n}\mathbb{E}\left[1_{\{|S_t-K|\leq\frac{1}{n}\}}\right].
\end{aligned}$$

But

$$\begin{aligned}
\mathbb{E}\left[1_{\{|S_t-K|\leq\frac{1}{n}\}}\right] &\leq \mathbb{P}\left(S_t-K\geq-\frac{1}{n}\right) \\
&\leq \mathbb{P}\left(S_t-S_0\geq K-S_0-\frac{1}{n}\right).
\end{aligned}$$

There exists $N_2 \geq 0$ such that for all $n \geq N_2$,

$$\begin{aligned}
\mathbb{P}\left(S_t-S_0\geq K-S_0-\frac{1}{n}\right) &\leq \mathbb{P}\left(S_t-S_0\geq\frac{K-S_0}{2}\right) \\
&\leq \left(\frac{2}{K-S_0}\right)^2\mathbb{E}\left[(S_t-S_0)^2\right],
\end{aligned}$$

by the Bienaymé-Chebyshev inequality. Hence,

$$\frac{1}{t}\mathbb{E}\left[(S_t-K)^+\right]\geq\frac{1}{t}\left(\mathbb{E}\left[f_n(S_t)\right]-f_n(S_0)\right)-\frac{1}{n}\left(\frac{2}{K-S_0}\right)^2\frac{1}{t}\mathbb{E}\left[\phi(S_t)-\phi(S_0)\right],$$

with $\phi(x)=(x-S_0)^2$. Applying Theorem 4.1 yields

$$\liminf_{t\to 0^+}\frac{1}{t}\,e^{-\int_0^t r(s)\,ds}\,\mathbb{E}\left[(S_t-K)^+\right]\geq\mathcal{L}_0 f_n(S_0)-\frac{1}{n}\left(\frac{2}{K-S_0}\right)^2\mathcal{L}_0\phi(S_0).$$

Letting $n\to+\infty$,

$$\lim_{t\to 0^+}\frac{1}{t}\,e^{-\int_0^t r(s)\,ds}\,\mathbb{E}\left[(S_t-K)^+\right]=\lim_{n\to\infty}\mathcal{L}_0 f_n(S_0).$$

Since $S_0<K$, $f_n=0$ in a neighborhood of S_0 for $n\geq N_1$ so $f_n(S_0)=f_n''(S_0)=f_n'(S_0)=0$ and $\mathcal{L}_0 f_n(S_0)$ reduces to

$$\mathcal{L}_0 f_n(S_0)=\int_{\mathbb{R}}[f_n(S_0e^y)-f_n(S_0)]m(0,dy).$$

A dominated convergence argument then yields

$$\lim_{n\to\infty} \mathcal{L}_0 f_n(S_0) = \int_{\mathbb{R}} [(S_0 e^y - K)_+ - (S_0 - K)_+] m(0, dy).$$

Using integration by parts, this last expression may be rewritten [14, Lemma 1] as

$$S_0 \psi_0 \left(\ln \left(\frac{K}{S_0} \right) \right)$$

where ψ_0 is given by (4.26). This ends the proof.

$\square$

Remark 4.4. *Theorem 4.2 also applies to in-the-money options, with a slight modification: for $K < S_{t_0}$,*

$$\frac{1}{t - t_0} \left(C_{t_0}(t, K) - (S_{t_0} - K) \right) \underset{t \to t_0^+}{\longrightarrow} r(t_0)\, S_{t_0} + S_{t_0} \psi_{t_0} \left(\ln \left(\frac{K}{S_{t_0}} \right) \right), \quad (4.30)$$

where

$$\psi_{t_0}(z) = \int_{-\infty}^{z} dx\, e^x \int_{-\infty}^{x} m(t_0, du), \qquad \text{for} \quad z < 0 \quad (4.31)$$

denotes the exponential double tail of $m(0, .)$.

4.3.2 At-the-money call options

When $S_{t_0} = K$, Theorem 4.2 does not apply. Indeed, as already noted in the case of Lévy processes by Tankov [91] and Figueroa-Lopez and Forde [43], the short maturity behavior of at-the-money options depends on whether a continuous martingale component is present and, in absence of such a component, on the degree of activity of small jumps, measured by the Blumenthal-Getoor index of the Lévy measure which measures its singularity at zero [60]. We will show here that similar results hold in the semimartingale case. We distinguish three cases:

1. S is a pure jump process of finite variation: in this case at-the-money call options behave linearly in $t - t_0$ (Proposition 4.4).

2. S is a pure jump process of infinite variation and its small jumps resemble those of an α-stable process: in this case at-the-money call options have an asymptotic behavior of order $|t - t_0|^{1/\alpha}$ when $t - t_0 \to 0^+$ (Proposition 4.5).

3. S has a continuous martingale component which is non-degenerate in the neighborhood of t_0: in this case at-the-money call options are of order $\sqrt{t-t_0}$ as $t \to t_0^+$, whether or not jumps are present (Theorem 4.3).

These statements are made precise in the sequel. We observe that, contrarily to the case of out-of-the money options where the presence of jumps dominates the asymptotic behavior, for at-the-money options the presence or absence of a continuous martingale (Brownian) component dominates the asymptotic behavior.

For the finite variation case, we use a slightly modified version of Assumption 4.5:

Assumption 4.7 (Weak right-continuity of jump compensator)**.** *For all* $\varphi \in \mathcal{C}_0^b(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$,

$$\lim_{t \to t_0,\, t > t_0} \mathbb{E}\left[\int_{\mathbb{R}} \left(e^{2y} \wedge |y|\right) \varphi(S_t, y)\, m(t, dy) | \mathcal{F}_{t_0}\right] = \int_{\mathbb{R}} \left(e^{2y} \wedge |y|\right) \varphi(S_{t_0}, y)\, m(t_0, dy).$$

Proposition 4.4 (Asymptotic for ATM call options for pure jump processes of finite variation)**.** *Consider the process*

$$S_t = S_0 + \int_0^t r(s) S_{s-}\, ds + \int_0^t \int_{-\infty}^{+\infty} S_{s^-}(e^y - 1) \tilde{M}(ds\ dy). \tag{4.32}$$

Under the Assumptions 4.7 and 4.6 and the condition,

$$\begin{aligned} &\forall t \in [t_0, T], \quad \int_{\mathbb{R}} |y|\, m(t, dy) < \infty, \\ &\frac{1}{t - t_0}\, C_{t_0}(t, S_{t_0}) \underset{t \to t_0^+}{\longrightarrow} \\ &\quad 1\left\{r(t_0) > \int_{\mathbb{R}} (e^y - 1) m(t_0, dy)\right\} S_{t_0} \left(r(t_0) + \int_{\mathbb{R}} (1 - e^y)^+ m(t_0, dy)\right) \\ &\quad + 1\left\{r(t_0) \leq \int_{\mathbb{R}} (e^y - 1) m(t_0, dy)\right\} S_{t_0} \int_{\mathbb{R}} (e^y - 1)^+\, m(t_0, dy). \end{aligned} \tag{4.33}$$

Proof. Replacing $\mathbb{P}$ by the conditional probability $\mathbb{P}_{\mathcal{F}_{t_0}}$, we may set $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all

$\mathbb{P}$-null sets. The Tanaka-Meyer formula applied to $(S_t - S_0)^+$ gives

$$\begin{aligned}
(S_t - S_0)^+ &= \int_0^t ds\, 1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1)\, m(s, dy) \right) \\
&+ \sum_{0 < s \le t} (S_s - S_0)^+ - (S_{s-} - S_0)^+.
\end{aligned}$$

Hence, applying Fubini's theorem,

$$\begin{aligned}
\mathbb{E}\left[(S_t - S_0)^+\right] &= \mathbb{E}\left[\int_0^t ds\, 1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy) \right)\right] \\
&+ \mathbb{E}\left[\int_0^t \int_{\mathbb{R}} \left[(S_{s-} e^y - S_0)^+ - (S_{s-} - S_0)^+\right]\, m(s, dy)\, ds\right] \\
&= \int_0^t ds\, \mathbb{E}\left[1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy) \right)\right] \\
&+ \int_0^t ds\, \mathbb{E}\left[\int_{\mathbb{R}} \left[(S_s e^y - S_0)^+ - (S_s - S_0)^+\right]\, m(s, dy)\right].
\end{aligned}$$

Furthermore, under the Assumptions 4.1 and 4.2 for $X_t = \log(S_t)$ (see equation (4.27)), one may apply Theorem 4.1 to the function

$$f : x \in \mathbb{R} \mapsto \exp(x),$$

yielding

$$\lim_{t \to 0^+} \frac{1}{t} \mathbb{E}\left[S_t - S_0\right] = S_0 \left(r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy) \right).$$

Furthermore, observing that

$$S_t\, 1_{\{S_t > S_0\}} = (S_t - S_0)^+ + S_0\, 1_{\{S_t > S_0\}},$$

we write

$$\begin{aligned}
\mathbb{E}\left[S_t\, 1_{\{S_t > S_0\}}\right] &= \mathbb{E}\left[(S_t - S_0)^+ + S_0\, 1_{\{S_t > S_0\}}\right] \\
&\le \mathbb{E}\left[|S_t - S_0|\right] + S_0\, \mathbb{P}\left(S_t > S_0\right),
\end{aligned}$$

using the Lipschitz continuity of $x \mapsto (x - S_0)_+$. Since $t \to \mathbb{E}[S_t]$ is right-continuous at 0, for t small enough :

$$0 \le \mathbb{E}\left[S_t\, 1_{\{S_t > S_0\}}\right] \le \frac{1}{2} + S_0.$$

Thus,

$$\begin{aligned}
&\lim_{t\to 0^+} \frac{1}{t}\mathbb{E}\left[\int_0^t ds\, 1_{\{S_{s-} > S_0\}} S_{s-} \left(r(s) - \int_{\mathbb{R}} (e^y - 1) m(s, dy)\right)\right] \\
= \;& S_0 \left(r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy)\right) 1\left\{r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy) > 0\right\}.
\end{aligned}$$

Let us now focus on the jump term and show that

$$t \in [0, T[\mapsto \mathbb{E}\left[\int_{\mathbb{R}} \left[(S_t e^y - S_0)^+ - (S_t - S_0)^+\right]\, m(t, dy)\right],$$

is right-continuous at 0 with right-limit

$$S_0 \int_{\mathbb{R}} (e^y - 1)^+\, m(0, dy).$$

One shall simply observes that

$$\left|(x e^y - S_0)^+ - (x - S_0)^+ - (S_0 e^y - S_0)^+\right| \le (x + S_0)\, |e^y - 1|,$$

using the Lipschitz continuity of $x \mapsto (x - S_0)_+$ and apply Assumption 4.7. To finish the proof, one shall gather both limits together :

$$\begin{aligned}
= \;& S_0 \left(r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy)\right) 1\left\{r(0) - \int_{\mathbb{R}} (e^y - 1) m(0, dy) > 0\right\} \\
+ \;& S_0 \int_{\mathbb{R}} (e^y - 1)^+\, m(0, dy).
\end{aligned}$$

This ends the proof.

□

Proposition 4.5 (Asymptotics of ATM call options for pure-jump martingales of infinite variation)**.** *Consider a semimartingale whose continuous martingale part is zero:*

$$S_t = S_0 + \int_0^t r(s) S_{s-}\, ds + \int_0^t \int_{-\infty}^{+\infty} S_{s^-}(e^y - 1) \tilde{M}(ds\ dy). \tag{4.34}$$

Under the Assumptions 4.5 and 4.6, if there exists $\alpha \in]1,2[$ and a family $m^\alpha(t,dy)$ of positive measures such that

$$\forall t \in [t_0,T], \quad m(\omega,t,dy) = m^\alpha(\omega,t,dy) + 1_{|y|\le 1}\frac{c(y)}{|y|^{1+\alpha}}\,dy \text{ a.s.}, \tag{4.35}$$

where $c(.) > 0$ is continuous at 0 and

$$\forall t \in [t_0,T] \quad \int_{\mathbb{R}} |y|\, m^\alpha(t,dy) < \infty, \tag{4.36}$$

then

$$\frac{1}{(t-t_0)^{1/\alpha}}\, C_{t_0}(t,S_{t_0}) \underset{t\to t_0^+}{\longrightarrow} \quad S_{t_0}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-c(0)\ |z|^\alpha}-1}{z^2}\,dz. \tag{4.37}$$

Proof. Without loss of generality, we set $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. The at-the-money call price can be expressed as

$$C_0(t,S_0) = \mathbb{E}\left[(S_t - S_0)^+\right] = S_0\mathbb{E}\left[\left(\frac{S_t}{S_0}-1\right)^+\right]. \tag{4.38}$$

Define, for $f \in C_b^2(]0,\infty[,\mathbb{R})$

$$\mathcal{L}_0 f(x) = r(0)xf'(x) + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y-1).f'(x)] m(0,dy). \tag{4.39}$$

We decompose $\mathcal{L}_0$ as the sum $\mathcal{L}_0 = \mathcal{K}_0 + \mathcal{J}_0$ where

$$\mathcal{K}_0 f(x) = r(0)xf'(x) + \int_{\mathbb{R}} [f(xe^y) - f(x) - x(e^y-1).f'(x)]\, m^\alpha(0,dy),$$
$$\mathcal{J}_0 f(x) = \int_{-1}^{1} [f(xe^y) - f(x) - x(e^y-1).f'(x)]\, \frac{c(y)}{|y|^{1+\alpha}}\,dy.$$

The term $\mathcal{K}_0$ may be be interpreted in terms of Theorem 4.1: if $(Z_t)_{[0,T]}$ is a *finite variation* semimartingale of the form (4.34) starting from $Z_0 = S_0$ with jump compensator $m^\alpha(t,dy)$, then by Theorem 4.1,

$$\forall f \in C_b^2(]0,\infty[,\mathbb{R}), \qquad \lim_{t\to 0^+} \frac{1}{t}\, e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[f(Z_t)\right] = \mathcal{K}_0 f(S_0). \tag{4.40}$$

The idea is now to interpret $\mathcal{L}_0 = \mathcal{K}_0 + \mathcal{J}_0$ in terms of a *multiplicative decomposition* $S_t = Y_t Z_t$ where $Y = \mathcal{E}(L)$ is the stochastic exponential of a pure-jump Lévy process with Lévy measure $c(y)/|y|^{1+\alpha}\,dy$, which we can take independent from Z. Indeed, let $Y = \mathcal{E}(L)$ where L is a pure-jump Lévy martingale with Lévy measure $1_{|y|\leq 1}\; c(y)/|y|^{1+\alpha}\; dy$, independent from Z, with infinitesimal generator $\mathcal{J}_0$. Then Y is a martingale and $[Y,Z]=0$. Then $S = YZ$ and Y is an exponential Lévy martingale, independent from Z, with $E[Y_t]=1$.

A result of Tankov [91, Proposition 5, Proof 2] for exponential Lévy processes then implies that

$$\frac{1}{t^{1/\alpha}}\,\mathbb{E}\left[(Y_t-1)^+\right] \overset{t\to 0^+}{\to} \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-c(0)|z|^\alpha}-1}{z^2}\,dz. \tag{4.41}$$

We will show that the term (4.41) is the dominant term which gives the asymptotic behavior of $C_0(T,S_0)$.

Indeed, by the Lipschitz continuity of $x \mapsto (x-S_0)_+$,

$$|(S_t-S_0)_+ - S_0(Y_t-1)_+| \leq Y_t|Z_t-S_0|,$$

so, taking expectations and using that Y is independent from Z, we get

$$\underbrace{\mathbb{E}[e^{-\int_0^t r(s)\,ds}|(S_t-S_0)_+}_{C_0(t,S_0)} - S_0(Y_t-1)_+|] \leq \underbrace{\mathbb{E}(Y_t)}_{=1}\,\mathbb{E}[e^{-\int_0^t r(s)\,ds}|Z_t-S_0|].$$

To estimate the right hand side of this inequality note that $|Z_t - S_0| = (Z_t-S_0)_+ + (S_0-Z_t)_+$. Since Z has finite variation, from Proposition 4.4

$$E[e^{-\int_0^t r(s)\,ds}(Z_t-S_0)_+] \overset{t\to 0^+}{\sim} tS_0\int_0^\infty dx\; e^x m([x,+\infty[).$$

Using the martingale property of $e^{-\int_0^t r(s)\,ds}Z_t)$ yields

$$E[e^{-\int_0^t r(s)\,ds}(S_0-Z_t)_+] \overset{t\to 0^+}{\sim} tS_0\int_0^\infty dx\; e^x m([x,+\infty[).$$

Hence, dividing by $t^{1/\alpha}$ and taking $t\to 0^+$ we obtain

$$\frac{1}{t^{1/\alpha}}\,e^{-\int_0^t r(s)\,ds}\,\mathbb{E}\left[|Z_t-S_0|^+\right] \overset{t\to 0^+}{\to} 0.$$

Thus, dividing by $t^{1/\alpha}$ the above inequality and using (4.41) yields

$$\frac{1}{t^{1/\alpha}}\, e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[(S_t - S_0)_+\right] \overset{t\to 0^+}{\to} S_0\, \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-c(0)|z|^\alpha} - 1}{z^2}\, dz.$$

□

We now focus on a third case, when S is a continuous semimartingale, i.e. an Ito process. From known results in the diffusion case [16], we expect in this case a short-maturity behavior ins $O(\sqrt{t})$. We propose here a proof of this behavior in a semimartingale setting using the notion of semimartingale local time.

Proposition 4.6 (Asymptotic for at-the-money options for continuous semimartingales). *Consider the process*

$$S_t = S_0 + \int_0^t r(s) S_s ds + \int_0^t S_s \delta_s dW_s. \tag{4.42}$$

Under the Assumptions 4.5 and 4.6 and the following non-degeneracy condition in the neighborhood of t_0*,*

$$\exists \epsilon > 0, \qquad \mathbb{P}\left(\forall t \in [t_0, T], \qquad \delta_t \geq \epsilon\right) = 1,$$

we have

$$\frac{1}{\sqrt{t - t_0}}\, C_{t_0}(t, S_{t_0}) \underset{t \to t_0^+}{\longrightarrow} \frac{S_{t_0}}{\sqrt{2\pi}}\, \delta_{t_0}. \tag{4.43}$$

Proof. Set $t_0 = 0$ and consider, without loss of generality, the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets. Applying the Tanaka-Meyer formula to $(S_t - S_0)^+$, we have

$$(S_t - S_0)^+ = \int_0^t 1_{\{S_s > S_0\}} dS_s + \frac{1}{2} L_t^{S_0}(S).$$

where $L_t^{S_0}(S)$ corresponds to the semimartingale local time of S_t at level S_0 under $\mathbb{P}$. As noted in Section 4.3.1, Assumption 4.6 implies that the discounted price $\hat{S}_t = e^{-\int_0^t r(s)\,ds} S_t$ is a $\mathbb{P}$-martingale. So

$$dS_t = e^{\int_0^t r(s)\,ds} \left(r(t) S_t dt + d\hat{S}_t \right), \quad \text{and}$$

$$\int_0^t 1_{\{S_s>S_0\}} dS_s = \int_0^t e^{\int_0^s r(u)\,du}\, 1_{\{S_s>S_0\}} d\hat{S}_s + \int_0^t e^{\int_0^s r(u)\,du}\, r(s) S_s 1_{\{S_s>S_0\}} ds,$$

where the first term is a martingale. Taking expectations, we get:

$$C(t,S_0) = \mathbb{E}\left[e^{-\int_0^t r(s)\,ds} \int_0^t e^{\int_0^s r(u)\,du}\; r(s) S_s\, 1_{\{S_s>S_0\}} ds + \frac{1}{2} e^{-\int_0^t r(s)\,ds}\, L_t^{S_0}(S) \right].$$

Since $\hat{S}$ is a martingale,

$$\forall t \in [0,T] \quad \mathbb{E}\left[S_t\right] = e^{\int_0^t r(s)\,ds}\, S_0 < \infty. \tag{4.44}$$

Hence $t \to \mathbb{E}\left[S_t\right]$ is right-continuous at 0:

$$\lim_{t\to 0^+} \mathbb{E}\left[S_t\right] = S_0. \tag{4.45}$$

Furthermore, under the Assumptions 4.1 and 4.2 for $X_t = \log(S_t)$ (see equation (4.27)), one may apply Theorem 4.1 to the function

$$f : x \in \mathbb{R} \mapsto (\exp(x) - S_0)^2,$$

yielding

$$\lim_{t\to 0^+} \frac{1}{t}\, \mathbb{E}\left[(S_t - S_0)^2\right] = \mathcal{L}_0 f(X_0),$$

where $\mathcal{L}_0$ is defined via equation (4.5) with $m \equiv 0$. Since $\mathcal{L}_0 f(X_0) < \infty$, then in particular,

$$t \mapsto \mathbb{E}\left[(S_t - S_0)^2\right]$$

is right-continuous at 0 with right limit 0. Observing that

$$S_t\, 1_{\{S_t>S_0\}} = (S_t - S_0)^+ + S_0\, 1_{\{S_t>S_0\}},$$

we write

$$\begin{aligned} \mathbb{E}\left[S_t\, 1_{\{S_t>S_0\}}\right] &= \mathbb{E}\left[(S_t - S_0)^+ + S_0\, 1_{\{S_t>S_0\}}\right] \\ &\le \mathbb{E}\left[|S_t - S_0|\right] + S_0\, \mathbb{P}\left(S_t > S_0\right), \end{aligned}$$

using the Lipschitz continuity of $x \mapsto (x - S_0)_+$. Since $t \to \mathbb{E}\left[S_t\right]$ is right-continuous at 0, for t small enough :

$$0 \le \mathbb{E}\left[S_t\, 1_{\{S_t>S_0\}}\right] \le \frac{1}{2} + S_0.$$

Thus, applying Fubini,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du}\, r(s) S_s\, 1_{\{S_s > S_0\}} ds\right] = O(t),$$

a fortiori,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du}\, r(s) S_s\, 1_{\{S_s > S_0\}} ds\right] = o\left(\sqrt{t}\right).$$

Hence (if the limit exists)

$$\lim_{t\to 0} \frac{1}{\sqrt{t}}\, C(t, S_0) = \lim_{t\to 0} \frac{1}{\sqrt{t}}\, e^{-\int_0^t r(s)\,ds}\, \mathbb{E}\left[\frac{1}{2}\, L_t^{S_0}(S)\right] = \lim_{t\to 0} \frac{1}{\sqrt{t}}\, \mathbb{E}\left[\frac{1}{2}\, L_t^{S_0}(S)\right]. \tag{4.46}$$

By the Dubins-Schwarz theorem [82, Theorem 1.5], there exists a Brownian motion B such that

$$\forall t < [U]_\infty, \quad U_t = \int_0^t \delta_s\, dW_s = B_{[U]_t} = B_{\int_0^t \delta_s^2 ds}.$$

$$\begin{aligned}
\text{So}\quad \forall t < [U]_\infty \quad S_t &= S_0 \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\delta_s^2\right) ds + B_{[U]_t}\right) \\
&= S_0 \exp\left(\int_0^t \left(r(s) - \frac{1}{2}\delta_s^2\right) ds + B_{\int_0^t \delta_s^2\, ds}\right).
\end{aligned}$$

The occupation time formula then yields, for $\phi \in \mathcal{C}_0^\infty(\mathbb{R}, \mathbb{R})$,

$$\begin{aligned}
\int_0^\infty \phi(K) L_t^K\left(S_0 \exp\left(B_{[U]}\right)\right) dK &= \int_0^t \phi\left(S_0 \exp\left(B_{[U]_u}\right)\right) S_0^2 \exp\left(B_{[U]_u}\right)^2 \delta_u^2\, du \\
&= \int_{-\infty}^\infty \phi(S_0 \exp(y)) S_0^2 \exp(y)^2 L_t^y\left(B_{[U]}\right) dy,
\end{aligned}$$

where $L_t^K\left(S_0 \exp\left(B_{[U]}\right)\right)$ (resp. $L_t^y\left(B_{[U]}\right)$) denotes the semimartingale local time of the process $S_0 \exp\left(B_{[U]}\right)$ at K and (resp. $B_{[U]}$ at y). A change of variable leads to

$$\begin{aligned}
&\int_{-\infty}^\infty \phi(S_0 \exp(y)) S_0 \exp(y) L_t^{S_0 e^y}\left(S_0 \exp\left(B_{[U]}\right)\right) dy \\
= &\int_{-\infty}^\infty \phi(S_0 \exp(y)) S_0^2 \exp(y)^2 L_t^y\left(B_{[U]}\right).
\end{aligned}$$

Hence
$$L_t^{S_0}\left(S_0 \exp\left(B_{[U]}\right)\right) = S_0 L_t^0\left(B_{[U]}\right).$$
We also have
$$L_t^0\left(B_{[U]}\right) = L^0_{\int_0^t \delta_s^2\, ds}(B),$$
where $L^0_{\int_0^t \delta_s^2\, ds}(B)$ denotes the semimartingale local time of B at time $\int_0^t \delta_s^2\, ds$ and level 0. Using the scaling property of Brownian motion,
$$\begin{aligned}
\mathbb{E}\left[L_t^{S_0}\left(S_0\, \exp\left(B_{[U]}\right)\right)\right] &= S_0\,\mathbb{E}\left[L^0_{\int_0^t \delta_s^2\, ds}(B)\right] \\
&= S_0\,\mathbb{E}\left[\sqrt{\int_0^t \delta_s^2\, ds}\, L_1^0(B)\right].
\end{aligned}$$
Hence
$$\begin{aligned}
\lim_{t\to 0^+}\frac{1}{\sqrt{t}}\,\mathbb{E}\left[L_t^{S_0}\left(S_0\, \exp\left(B_{[U]}\right)\right)\right] &= \lim_{t\to 0^+}\frac{1}{\sqrt{t}}\,S_0\,\mathbb{E}\left[\sqrt{\int_0^t \delta_s^2\, ds}\, L_1^0(B)\right] \\
&= \lim_{t\to 0^+}\, S_0\,\mathbb{E}\left[\sqrt{\frac{1}{t}\int_0^t \delta_s^2\, ds}\, L_1^0(B)\right].
\end{aligned}$$
Let us show that
$$\lim_{t\to 0^+}\, S_0\,\mathbb{E}\left[\sqrt{\frac{1}{t}\int_0^t \delta_s^2\, ds}\, L_1^0(B)\right] = S_0\,\delta_0\,\mathbb{E}\left[L_1^0(B)\right]. \tag{4.47}$$
Using the Cauchy-Schwarz inequality,
$$\left|\mathbb{E}\left[\left(\sqrt{\frac{1}{t}\int_0^t \delta_s^2\, ds} - \delta_0\right)\, L_1^0(B)\right]\right| \le \mathbb{E}\left[L_1^0(B)^2\right]^{1/2}\,\mathbb{E}\left[\left(\sqrt{\frac{1}{t}\int_0^t \delta_s^2\, ds} - \delta_0\right)^2\right]^{1/2}.$$
The Lipschitz property of $x \to \left(\sqrt{x} - \delta_0\right)^2$ on $[\epsilon, +\infty[$ yields
$$\begin{aligned}
\mathbb{E}\left[\left(\sqrt{\frac{1}{t}\int_0^t \delta_s^2\, ds} - \delta_0\right)^2\right] &\le c(\epsilon)\mathbb{E}\left[\left|\frac{1}{t}\int_0^t \left(\delta_s^2 - \delta_0^2\right)\, ds\right|\right] \\
&\le \frac{c(\epsilon)}{t}\int_0^t ds\,\mathbb{E}\left[\left|\delta_s^2 - \delta_0^2\right|\right].
\end{aligned}$$

where $c(\epsilon)$ is the Lipschitz constant of $x \to (\sqrt{x}-\delta_0)^2$ on $[\epsilon, +\infty[$. Assumption 4.5 and Lemma 4.1 then imply (4.47). By Lévy's theorem for the local time of Brownian motion, $L_1^0(B)$ has the same law as $|B_1|$, leading to

$$\mathbb{E}\left[L_1^0(B)\right] = \sqrt{\frac{2}{\pi}}.$$

Clearly, since $L_t^K(S) = L_t^K\left(S_0 \exp\left(B_{[U]}\right)\right)$,

$$\lim_{t\to 0} \frac{1}{\sqrt{t}}\,\mathbb{E}\left[\frac{1}{2}L_t^{S_0}(S)\right] = \frac{S_0}{\sqrt{2\pi}}\,\delta_0. \tag{4.48}$$

This ends the proof. $\square$

We can now treat the case of a general Itô semimartingale with both a continuous martingale component and a jump component.

Theorem 4.3 (Short-maturity asymptotics for at-the-money call options)**.** *Consider the price process S whose dynamics is given by*

$$S_t = S_0 + \int_0^t r(s)S_{s^-}ds + \int_0^t S_{s^-}\delta_s dW_s + \int_0^t \int_{-\infty}^{+\infty} S_{s^-}(e^y-1)\tilde{M}(ds\ dy).$$

Under the Assumptions 4.5 and 4.6 and the folllowing non-degeneracy condition in the neighborhood of t_0

$$\exists \epsilon > 0, \qquad \mathbb{P}\left(\forall t \in [t_0, T], \quad \delta_t \geq \epsilon\right) = 1,$$

we have

$$\frac{1}{\sqrt{t-t_0}}\,C_{t_0}(t, S_{t_0}) \underset{t\to t_0^+}{\longrightarrow} \frac{S_{t_0}}{\sqrt{2\pi}}\,\delta_{t_0}. \tag{4.49}$$

Proof. Applying the Tanaka-Meyer formula to $(S_t - S_0)^+$, we have

$$\begin{aligned}(S_t - S_0)^+ = &\int_0^t 1_{\{S_{s-}>S_0\}} dS_s + \frac{1}{2}L_t^{S_0} \\ &+ \sum_{0<s\leq t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-}>S_0\}}\Delta S_s.\end{aligned} \tag{4.50}$$

As noted above, Assumption 4.6 implies that the discounted price $\hat{S}_t = e^{-\int_0^t r(s)\,ds} S_t$ is a martingale under $\mathbb{P}$. So (S_t) can be expressed as $dS_t = e^{\int_0^t r(s)\,ds}\left(r(t)S_{t-}dt + d\hat{S}_t\right)$ and

$$\int_0^t 1_{\{S_{s-}>S_0\}} dS_s = \int_0^t e^{\int_0^s r(u)\,du}\, 1_{\{S_{s-}>S_0\}} d\hat{S}_s + \int_0^t e^{\int_0^s r(u)\,du}\, r(s) S_{s-} 1_{\{S_{s-}>S_0\}} ds,$$

where the first term is a martingale. Taking expectations, we get

$$\begin{aligned} e^{\int_0^t r(s)\,ds} C(t, S_0) &= \mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du}\, r(s) S_s\, 1_{\{S_{s-}>S_0\}} ds + \frac{1}{2} L_t^{S_0}\right] \\ &+ \mathbb{E}\left[\sum_{0<s\le t} (S_s - S_0)^+ - (S_{s-} - S_0)^+ - 1_{\{S_{s-}>S_0\}} \Delta S_s\right]. \end{aligned}$$

Since $\hat{S}$ is a martingale,

$$\forall t \in [0, T] \quad \mathbb{E}\left[S_t\right] = e^{\int_0^t r(s)\,ds}\, S_0 < \infty. \tag{4.51}$$

Hence $t \to \mathbb{E}\left[S_t\right]$ is right-continuous at 0:

$$\lim_{t\to 0^+} \mathbb{E}\left[S_t\right] = S_0. \tag{4.52}$$

Furthermore, under the Assumptions 4.1 and 4.2 for $X_t = \log(S_t)$ (see equation (4.27)), one may apply Theorem 4.1 to the function

$$f : x \in \mathbb{R} \mapsto (\exp(x) - S_0)^2,$$

yielding

$$\lim_{t\to 0^+} \frac{1}{t}\, \mathbb{E}\left[(S_t - S_0)^2\right] = \mathcal{L}_0 f(X_0),$$

where $\mathcal{L}_0$ is defined via equation (4.5). Since $\mathcal{L}_0 f(X_0) < \infty$, then in particular,

$$t \mapsto \mathbb{E}\left[(S_t - S_0)^2\right]$$

is right-continuous at 0 with right limit 0. Observing that

$$S_t\, 1_{\{S_t>S_0\}} = (S_t - S_0)^+ + S_0\, 1_{\{S_t>S_0\}},$$

we write

$$\begin{aligned}\mathbb{E}\left[S_t\, 1_{\{S_t>S_0\}}\right] &= \mathbb{E}\left[(S_t-S_0)^+ + S_0\, 1_{\{S_t>S_0\}}\right]\\ &\le \mathbb{E}\left[|S_t-S_0|\right] + S_0\,\mathbb{P}\left(S_t>S_0\right),\end{aligned}$$

using the Lipschitz continuity of $x \mapsto (x-S_0)_+$. Since $t \to \mathbb{E}[S_t]$ is right-continuous at 0, for t small enough :

$$0 \le \mathbb{E}\left[S_t\, 1_{\{S_t>S_0\}}\right] \le \frac{1}{2} + S_0.$$

Thus, applying Fubini,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du}\, r(s) S_s\, 1_{\{S_s>S_0\}} ds\right] = O(t),$$

a fortiori,

$$\mathbb{E}\left[\int_0^t e^{\int_0^s r(u)\,du}\, r(s) S_s\, 1_{\{S_s>S_0\}} ds\right] = o\left(\sqrt{t}\right).$$

Let us now focus on the jump part,

$$\begin{aligned}&\mathbb{E}\left[\sum_{0<s\le t} (S_s-S_0)^+ - (S_{s-}-S_0)^+ - 1_{\{S_{s-}>S_0\}}\Delta S_s\right]\\ =\ &\mathbb{E}\left[\int_0^t ds \int m(s,dx)\,(S_{s-}e^x-S_0)^+ - (S_{s-}-S_0)^+ - 1_{\{S_{s-}>S_0\}} S_{s-}(e^x-1)\right]\end{aligned} \tag{4.53}$$

Observing that

$$\left|(ze^x-S_0)^+ - (z-S_0)^+ - 1_{\{z>S_0\}} z(e^x-1)\right| \le C\,(S_0e^x-z)^2,$$

then, together with Assumption 4.5 and Lemma 4.1 implies,

$$\mathbb{E}\left[\sum_{0<s\le t} (S_s-S_0)^+ - (S_{s-}-S_0)^+ - 1_{\{S_{s-}>S_0\}}\Delta S_s\right] = O(t) = o(\sqrt{t}).$$

Since $\delta_0 \ge \epsilon$, equation (4.48) yields the result. □

Remark 4.5. *As noted by Berestycki et al [17, 18] in the diffusion case, the regularity of f at S_{t_0} plays a crucial role in the asymptotics of $\mathbb{E}[f(S_t)]$. Theorem 4.1 shows that $\mathbb{E}[f(S_t)] \sim ct$ for smooth functions f, even if $f(S_{t_0}) = 0$, while for call option prices we have $\sim \sqrt{t}$ asymptotics at-the-money where the function $x \to (x - S_0)_+$ is not smooth.*

Remark 4.6. *In the particular case of a Lévy process, Proposition 4.4, Proposition 4.5 and Theorem 4.3 imply [91, Proposition 5, Proof 2].*

Chapter 5

Application to index options in a jump-diffusion model

5.1 Introduction

Consider a multi-asset market with d assets, whose prices $S^1, ..., S^d$ are represented as Ito semimartingales:

$$S_t^i = S_0^i + \int_0^t r(s) S_{s^-}^i ds + \int_0^t S_{s^-}^i \delta_s^i dW_s^i + \int_0^t \int_{\mathbb{R}^d} S_{s^-}^i (e^{y_i} - 1) \tilde{N}(ds\, dy),$$

where

- δ^i is an adapted process taking values in $\mathbb{R}$ representing the volatility of the asset i, W is a d-dimensional Wiener process : for all $1 \leq (i,j) \leq d$, $\langle W^i, W^j \rangle_t = \rho_{ij} t$,
- N is a Poisson random measure on $[0,T] \times \mathbb{R}^d$ with compensator $\nu(dy)\, dt$,
- $\tilde{N}$ denotes its compensated random measure.

We consider an index, defined as a weighted sum of the asset prices:

$$I_t = \sum_{i=1}^d w_i S_t^i, \quad d \geq 2.$$

The pricing of index options involves the computation of quantities of the form $\mathbb{E}\left[f(I_t) | \mathcal{F}_{t_0}\right]$ and Chapter 4 shows that the short time asymptotics for these quantitie that we have characterized explicitly in terms of the characteristic triplet of the discontinuous semimartingale I_t in Chapter 3.

Short time asymptotics of index call option prices have been computed by Avellaneda & al [5] in the case where S is a continuous process. Results of Chapter 4 show that this asymptotic behavior is quite diferent for at the money or out of the money options. At the money options exhibit a bahavior in $O(\sqrt{t})$ which involves the diffusion component of I_t whereas out of the money options exhibit a linear behavior in t which only involves the jumps of I_t.

In this Chapter, we propose an analytical approximation for short maturity index options, generalizing the approach by Avellaneda & al. [5] to the multivariate jump-diffusion case. We implement this method in the case of the Merton model in dimension $d = 2$ and $d = 30$ and study its numerical precision.

The main difficulty is that, even when the joint dynamics of the index components $(S^1, ..., S^d)$ is Markovian, the index I_t is not a Markov process but only a semimartingale. The idea is to consider the *Markovian projection* of the index process, an auxiliary Markov process which has the same marginals as I_t, and use it to derive the asymptotics of index options, using the results of Chapter 4. This approximation is shown to depend only on the coefficients of this Markovian projection, so the problem boils down to computing effectively these coefficients: the local volatility function and the 'effective Lévy measure', defined as the conditional expectation given I_t of the jump compensator of I.

The computation of the effective Lévy measure involves a d-dimensional integral. Computing directly this integral would lead, numerically speaking, to a complexity increasing exponentially with d. We propose different techniques to simplify this computation and make it feasible when the dimension is large, using the Laplace method to approximate the exponential double tail of the jump measure of I_t. Laplace method is an important tool when one wants to approximate consistently high-dimensional integrals and avoids a numerical exponential complexity increasing with the dimension. Avellaneda & al [5] use this method in the diffusion case to compute the local volatility of an index option by using a steepest descent approximation, that is by considering that, for t small enough, the joint law of $(S^1, \cdots, S^d)$ given

$$\{\sum_{i=1}^{d} w_i S_t^i = u\},$$

is concentrated around the most probable path, which we proceed to identify.

5.2 Short maturity asymptotics for index options

We recall that the value $C_{t_0}(t,K)$ at time t_0 of an index call option with expiry $t > t_0$ and strike $K > 0$ is given by

$$C_{t_0}(t,K) = e^{-\int_{t_0}^t r(s)\,ds}\,\mathbb{E}^{\mathbb{P}}\left[\max(I_t - K, 0)|\mathcal{F}_{t_0}\right]. \tag{5.1}$$

Replacing $\mathbb{P}$ by the conditional measure $\mathbb{P}_{|\mathcal{F}_{t_0}}$ given $\mathcal{F}_{t_0}$, we may replace the conditional expectation in (5.1) by an expectation with respect to the marginal distribution of I_t under $\mathbb{P}_{|\mathcal{F}_{t_0}}$. Thus, we put $t_0 = 0$ in the sequel and consider the case where $\mathcal{F}_0$ is the σ-algebra generated by all $\mathbb{P}$-null sets:

$$C_0(t,K) = e^{-\int_0^t r(s)\,ds}\,\mathbb{E}^{\mathbb{P}}\left[\max(I_t - K, 0)\right]. \tag{5.2}$$

In Chapter 3, we have shown that one can characterize in an explicit manner the characteristic triplet of the semimartingale (I_t). Namely, let us make the following assumptions:

Assumption 5.1 (Right-continuity at 0)**.**

$$\lim_{t\to 0^+} \mathbb{E}\left[\|\delta_t - \delta_0\|^2\right] = 0.$$

Assumption 5.2 (Integrability condition)**.**

$$\mathbb{E}\left[\exp\left(\frac{1}{2}\int_0^t \|\delta_s\|^2\,ds\right)\right] < \infty, \qquad \int_{\mathbb{R}^d} \left(\|y\| \wedge \|y\|^2\right)\,\nu(dy) < \infty.$$

Set

$$\forall 1 \le i \le d, \quad \alpha_t^i = \frac{w_i S_t^i}{I_t}, \qquad \sum_{i=1}^d \alpha_t^i = 1, \tag{5.3}$$

and define

$$\begin{aligned} \sigma_t &= \left(\sum_{i,j=1}^d \alpha_t^i \alpha_t^j \rho_{ij}\,\delta_t^i \delta_t^j\right)^{\frac{1}{2}}, \\ \forall x \ge 0, \quad \varphi_t(x) &= \int_{\{z\in\mathbb{R}^d - \{0\}, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{z_i}\right)\ge x\}} \nu(dz), \end{aligned} \tag{5.4}$$

and similarly for $x < 0$. Then under Assumption 5.2, the proof of Theorem 3.3 implies,

$$I_t = I_0 + \int_0^t r(s) I_{s-}\, ds + \int_0^t \sigma_s\, I_{s-}\, dB_s + \int_0^t \int_{\mathbb{R}^d} (e^u - 1)\, I_{s-} \tilde{K}(ds\, du), \quad (5.5)$$

where B_t is a Brownian motion, K (resp. $\tilde{K}$) is an integer-valued random measure on $[0,T] \times \mathbb{R}$ (resp. its compensated random measure.) with compensator $k(t, du)\, dt$ defined via its tail φ_t.

We derive the short-maturity asymptotics for the index call options (at-the money and out-of-the money) : we apply Theorem 4.2 and Theorem 4.3 to the one-dimensional semimartingale I_t (see Chapter 4).

Theorem 5.1 (Short time asymptotics of index call options)**.** *Under the Assumptions 5.1 and 5.2, if there exists $\gamma > 0$, $\|\delta_t\| > \gamma$ for all $t \in [0,T]$ almost surely, then*

1.

$$\frac{1}{\sqrt{t}}\, C_0(t, I_0) \underset{t \to 0^+}{\longrightarrow} \frac{I_0}{\sqrt{2\pi}}\, \sigma_0, \quad (5.6)$$

2.

$$\forall K > I_0, \quad \frac{1}{t}\, C_0(t, K) \underset{t \to 0^+}{\longrightarrow} I_0\, \eta_0 \left(\ln \left(\frac{K}{I_0} \right) \right), \quad (5.7)$$

where

$$\eta_0(y) = \int_y^{+\infty} dx\ e^x \int_x^{\infty} k(0, du) \equiv \int_y^{+\infty} dx\ e^x\, \varphi_0(x), \quad (5.8)$$

denotes the exponential double tail of $k(0, du)$ for $y > 0$.

Proof. First, the volatility of I_t satisfies, for $t \in [0,T]$,

$$\sigma_t^2 = \sum_{i,j=1}^{d} \alpha_t^i \alpha_t^j \rho_{ij}\, \delta_t^i \delta_t^j \ \leq \sum_{i,j=1}^{d} \rho_{ij}\, \delta_t^i \delta_t^j.$$

Using the concavity property of the logarithm,

$$\ln \left(\sum_{1 \leq i \leq d} \alpha_t^i\, e^{y_i} \right) \geq \sum_{1 \leq i \leq d} \alpha_t^i\, y_i \geq -\|y\|$$

and

$$\ln\left(\sum_{1\le i\le d}\alpha_t\, e^{y_i}\right)\le \ln\left(\max_{1\le i\le d}\, e^{y_i}\right)\le \max_{1\le i\le d} y_i.$$

Thus,

$$\left|\ln\left(\sum_{1\le i\le d}\alpha_t^i\, e^{y_i}\right)\right|\le \|y\|.$$

Furthermore, clearly,

$$\sum_{1\le i\le d}\alpha_t^i e^{z_i}-1\le e^{\|z\|}-1,$$

and the convexity property of the exponential yields

$$\begin{aligned} e^{\|z\|}+\sum_{1\le i\le d}\alpha_t^i e^{z_i} &\ge e^{\|z\|}+\exp\left(\sum_{1\le i\le d}\alpha_t^i\, z_i\right)\\ &= e^{\|z\|}+e^{\alpha_t.z}\ge e^{\|z\|}+e^{-\|\alpha_t\|\|z\|}\ge e^{\|z\|}+e^{-\|z\|}\ge 2. \end{aligned}$$

Hence

$$\left(\sum_{1\le i\le d}\alpha_t^i e^{z_i}-1\right)^2\le \left(e^{\|z\|}-1\right)^2.$$

Thus,

$$\begin{aligned} &\frac{1}{2}\int_0^t \sigma_s^2\, ds+\int_0^t\int_{\mathbb{R}}(e^u-1)^2 k(s,\, du)\, ds\\ =\;& \frac{1}{2}\int_0^t\sum_{i,j=1}^d \alpha_s^i\alpha_s^j\rho_{ij}\,\delta_s^i\delta_s^j\, ds\\ +\;& \int_0^t\int_{\mathbb{R}^d}\left(\sum_{1\le i\le d}\alpha_s^i e^{z_i}-1\right)^2\nu(dz_1,\cdots,dz_d)\, ds\\ \le\;& \frac{1}{2}\int_0^t\sum_{i,j=1}^d \rho_{ij}\,\delta_s^i\delta_s^j\, ds+\int_0^t\int_{\mathbb{R}^d}(e^{\|z\|}-1)^2\nu(dz_1,\cdots,dz_d)\, ds<\infty. \end{aligned}$$

If Assumption 5.2 holds then I_t satisfies Assumption 4.6.

Let $\varphi \in \mathcal{C}_0^b(\mathbb{R}^+ \times \mathbb{R}, \mathbb{R})$. Applying Fubini's theorem yields

$$
\begin{aligned}
& \mathbb{E}\left[\int_{\mathbb{R}} (e^u - 1)^2 \; \varphi(I_t, u)\, k(t, du)\right] \\
= \; & \mathbb{E}\left[\int_{\mathbb{R}^d} \left(\sum_{1\le i\le d} \alpha_t^i e^{z_i} - 1\right)^2 \varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right))\, \nu(dz_1, \cdots, dz_d)\right] \\
= \; & \int_{\mathbb{R}^d} \mathbb{E}\left[\left(\sum_{1\le i\le d} \alpha_t^i e^{z_i} - 1\right)^2 \varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right))\right] \nu(dz_1, \cdots, dz_d).
\end{aligned}
$$

Let us show that

$$
\lim_{t\to 0^+} \mathbb{E}\left[\left(\sum_{1\le i\le d} \alpha_t^i\, e^{z_i} - 1\right)^2 \varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right))\right] \tag{5.9}
$$

$$
= \; \mathbb{E}\left[\left(\sum_{1\le i\le d} \alpha_0^i e^{z_i} - 1\right)^2 \varphi(I_0, \ln\left(\sum_{1\le i\le d} \alpha_0^i e^{y_i}\right))\right]. \tag{5.10}
$$

First, note that

$$
\begin{aligned}
& \left|\left(\sum_{1\le i\le d} \alpha_t^i\, e^{z_i} - 1\right)^2 \varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right)) - \left(\sum_{1\le i\le d} \alpha_0^i e^{z_i} - 1\right)^2 \varphi(I_0, \ln\left(\sum_{1\le i\le d} \alpha_0^i e^{y_i}\right))\right| \\
\le \; & \left|\left[\left(\sum_{1\le i\le d} \alpha_t^i\, e^{z_i} - 1\right)^2 - \left(\sum_{1\le i\le d} \alpha_0^i e^{z_i} - 1\right)^2\right] \varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right))\right| \\
+ \; & \left|\left(\sum_{1\le i\le d} \alpha_0^i\, e^{z_i} - 1\right)^2 \left(\varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right)) - \varphi(I_0, \ln\left(\sum_{1\le i\le d} \alpha_0^i e^{y_i}\right))\right)\right| \\
\le \; & 2\left(\sum_{1\le i\le d} e^{z_i} + 2\right) \|\varphi\|_\infty \sum_{1\le i\le d} |\alpha_t^i - \alpha_0^i|\, e^{z_i} \\
+ \; & \left(e^{\|z\|} - 1\right)^2 \left|\varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right)) - \varphi(I_0, \ln\left(\sum_{1\le i\le d} \alpha_0^i e^{y_i}\right))\right|.
\end{aligned}
$$

We recall that $\alpha_t^i = \frac{w_i S_t^i}{I_t}$. Since I, S^i's are càdlàg,

$$I_t \underset{t\to 0^+}{\longrightarrow} I_0,\ \alpha_t^i \underset{t\to 0^+}{\longrightarrow} \alpha_0^i,\ \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right) \underset{t\to 0^+}{\longrightarrow} \ln\left(\sum_{1\le i\le d} \alpha_0^i e^{y_i}\right) \quad \text{a.s.}.$$

Since $\sum_{1\le i\le d} |\alpha_t^i - \alpha_0^i|\, e^{z_i} \le \sum_{1\le i\le d} 2e^{z_i}$, and φ is bounded on $\mathbb{R}^+ \times \mathbb{R}$, a dominated convergence argument yields

$$\mathbb{E}\left[\sum_{1\le i\le d} |\alpha_t^i - \alpha_0^i|\, e^{z_i}\right] \underset{t\to 0^+}{\longrightarrow} 0,$$
$$\mathbb{E}\left[\left|\varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right)) - \varphi(I_0, \ln\left(\sum_{1\le i\le d} \alpha_0^i e^{y_i}\right))\right|\right] \underset{t\to 0^+}{\longrightarrow} 0.$$

Since

$$\mathbb{E}\left[\left(\sum_{1\le i\le d} \alpha_t^i e^{z_i} - 1\right)^2 \varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right))\right] \le \|\varphi\|_\infty \left(e^{\|z\|} - 1\right)^2$$
$$\mathbb{E}\left[\ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right)^2 \varphi(I_t, \ln\left(\sum_{1\le i\le d} \alpha_t^i e^{y_i}\right))\right] \le \|\varphi\|_\infty \|z\|^2,$$

a dominated convergence argument implies that

$$\lim_{t\to 0^+} \mathbb{E}\left[\int_{\mathbb{R}} (e^u - 1)^2\ \varphi(I_t, u)\, k(t, du)\right] = \int_{\mathbb{R}} (e^u - 1)^2\ \varphi(I_0, u)\, k(0, du).$$

For convenience, let us introduce $U_t = \left(\alpha_t^1 \delta_t^1, \cdots, \alpha_t^d \delta_t^d\right)$, $P = (\rho_{ij})_{1\le i,j\le d}$ then the volatility of I_t simply rewrites

$$\sigma_t = \langle U_t, P U_t\rangle^{1/2}.$$

The Lipschitz property of $x \to \left(\sqrt{x} - \sigma_0\right)^2$ on $[\gamma, +\infty[$ yields there exists $C > 0$,

$$|\sigma_t - \sigma_0|^2 \le C\ |\langle U_t, P U_t\rangle - \langle U_0, P U_0\rangle|\,.$$

The continuity of $< .,. >$ yields

$$\exists C' > 0, \quad |\langle U_t, P U_t\rangle - \langle U_0, P U_0\rangle| \le C' \max_{1\le i,j\le d} \rho_{i,j}\, \|U_t - U_0\|^2.$$

Since $0 < \alpha_t^i < 1$,

$$\|U_t - U_0\|^2 = \sum_{i=1}^d \left(\alpha_t^i\,\delta_t^i - \alpha_0^i\,\delta_0^i\right)^2 \;\leq\; \sum_{i=1}^d \left(\alpha_t^i\,\delta_t^i - \alpha_0^i\,\delta_0^i\right)^2 \;\leq\; \sum_{i=1}^d \left(\delta_t^i - \delta_0^i\right)^2,$$

Assumption 5.1 yields

$$\mathbb{E}\left[|\sigma_t - \sigma_0|^2\right] \underset{t\to 0^+}{\longrightarrow} 0.$$

Thus, Assumption 4.5 and Assumption 4.6 hold for σ_t and $k(t,.)$ and one shall apply Theorem 4.2 and Theorem 4.3 to I_t to yield the result. □

5.3 Example : the multivariate Merton model

We decide in the sequel to study the Merton model : each asset writes

$$dS_t^i = S_{t^-}^i \left(r\,dt + \delta^i dW_t^i + \left(e^{Y_i^{N_t}} - 1\right) d\tilde{N}(dt)\right), \tag{5.11}$$

where $\delta^i > 0$, W^i is a Wiener process, N_t is a Poisson process with intensity λ and the jumps $\left(Y_i^k\right)_{k\geq 1}$ are *iid* replications of a Gaussian random variable $Y_i^k \sim \mathcal{N}(m_i, v_i)$.

Empirical studies [27] shows that one can considers the case when the Wiener processes W^i and the jumps Y_i are homogeneously correlated, implying that the correlation matrix of the Wiener processes, denoted Σ^W, and the correlation matrix of the jumps, denoted Σ^J, are of the form

$$\Sigma^W = \begin{pmatrix} 1 & \cdots & \rho_W \\ \vdots & \ddots & \vdots \\ \rho_W & \cdots & 1 \end{pmatrix}, \quad \Sigma^J = \begin{pmatrix} 1 & \cdots & \rho_J \\ \vdots & \ddots & \vdots \\ \rho_J & \cdots & 1 \end{pmatrix}, \tag{5.12}$$

that is all the non-diagonal coefficients are the same. We choose the index to be equally weighted:

$$\forall 1 \leq i \leq d \quad w_i = \frac{1}{d}.$$

Considering the volatilities matrix V of the jumps,

$$V = \begin{pmatrix} v_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & v_d \end{pmatrix}, \tag{5.13}$$

then the covariation matrix Θ of the jumps writes

$$\Theta = V\,\Sigma^J\,V. \tag{5.14}$$

The compensator of the Poisson random measure describing the jump dynamic of the S_t^i's is denoted $\nu(dz) \equiv n(z)\,dz$ and is expressed as follows,

$$n(z) = \lambda\, g_{m,\Theta}(z), \tag{5.15}$$

with

$$g_{m,\Theta}(z) = \frac{1}{(2\pi)^{d/2}\,|\Theta|^{1/2}}\,\exp\left(-\frac{1}{2}{}^t(z-m)\Theta^{-1}(z-m)\right). \tag{5.16}$$

Clearly Assumption 5.1 and 5.2 hold in this framework since the multivariate gaussian distribution $g_{m,\Theta}(z)$ has finite moments for all orders and one may apply Theorem 5.1.

For a small maturity T, we propose the following approximations for $C_0(T,K)$:

1. At the money:

$$C_0(T,I_0) \simeq \sqrt{T}\frac{I_0}{\sqrt{2\pi}}\,\sigma_0, \tag{5.17}$$

with

$$\sigma_0 = \frac{1}{d}\left(\rho_W \sum_{i\neq j}^{d} \delta^i\delta^j\, S_0^i S_0^j + \sum_{i=1}^{d}\left(\delta^i\, S_0^i\right)^2\right)^{\frac{1}{2}}. \tag{5.18}$$

2. Out of the money:

$$\forall K > I_0 \quad C_0(T,K) \sim T\, I_0\, \eta_0\left(\ln\left(\frac{K}{I_0}\right)\right). \tag{5.19}$$

The approximation at the money case is simple in this situation since there is no numerical difficulty to compute σ_0. We will not study this case in the sequel. We refer to Avellaneda & al [5] for a complete study.

For the out of the money case, a more care is needed. Looking at equation (5.19), one has to compute the quantity

$$l(K) = I_0 \, \eta_0 \left(\ln \left(\frac{K}{I_0} \right) \right), \tag{5.20}$$

namely, $\eta_0(y)$ for $y > 1$ (observe that $\ln\left(\frac{K}{I_0}\right) > 1$ in the out of the money case) or $\varphi_0(x)$ for $x > 1$ since equation (5.8). Clearly it leads to the computation of an integral of dimension d in a certain region of $\mathbb{R}^d$,

$$z \in \mathbb{R}^d - \{0\} \quad \text{such that} \quad \ln \left(\sum_{1 \leq i \leq d} \alpha_0^i e^{z_i} \right) \geq x, \tag{5.21}$$

implying, numerically speaking, an exponential complexity increasing with the dimension d.

We present here alternative techniques (either probabilistic or analytic) to either compute or approximate $k(0,.)$, φ_0 or η_0. We distinguish two cases:

1. When the dimension d is small, we can compute φ_0 using (5.4) then η_0 via (5.8) via low-dimensional quadrature. The representation (5.4) offers us a probabilistic interpretation of φ_0. In the special case of the Merton model, since the measure

   $$\frac{1}{\lambda} \nu(dz) \equiv g_{m,\Theta}(z) \, dz$$

   corresponds to a multivariate gaussian distribution, let $Z = (Z_1, \ldots, Z_d)$ be a multivariate gaussian random variable with law $\mathcal{N}(m, \Theta)$. Then equation (5.4) simply rewrites

   $$\forall x > 0 \quad \varphi_0(x) = \mathbb{P}\left(\alpha_1 e^{Z_1} + \cdots + \alpha_d e^{Z_d} \geq \exp(x) \right). \tag{5.22}$$

   Using that $(Z_1, \cdots, Z_{d-1})$ is still a Gaussian vector, one oberves that the quantity (5.22) may be computed in an explicit manner. The d-dimensional integral involved in the computation of $\varphi_0(x)$, may be reduced to the computation of an integral of dimension $d-1$, and so on recursively. The particular case when $d = 2$ becomes simple and the computation is easy and exact. We decide to retain this approach for the dimension $d = 2$.

2. For higher dimensions, we propose an alternative approach. Integration by parts allows us to rewrite η_0 as,

$$\eta_0(y) = \int_y^{\infty} (e^u - e^y)\, k(0,u)\, du. \tag{5.23}$$

Let us define

$$g_{u,\epsilon}(z) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left(-\frac{1}{2\epsilon^2}\left(u - \ln\left(\sum_1^d \alpha_i e^{z_i}\right)\right)^2\right) \tag{5.24}$$

and

$$k_\epsilon(u) = \lambda \int_{\mathbb{R}^d} g_{u,\epsilon}(z)\, g_{m,\Theta}(z)\, dz. \tag{5.25}$$

Then given equation (5.4), under Assumption 5.2, a dominated convergence argument yields

$$\lim_{\epsilon \to 0} \int_y^{\infty} (e^u - e^y)\, k_\epsilon(u)\, du = \eta_0(y). \tag{5.26}$$

5.3.1 The two-dimensional case

In the two dimensional case, we can explicitly compute the exponential double tail using a probabilistic interpretation of $\eta_0(y)$. We recall that

$$\eta_0(y) = \int_y^{+\infty} dx\ e^x\, \varphi_0(x).$$

Since the jump measure

$$\frac{1}{\lambda}\nu(dz) \equiv g_{m,\Theta}(z)$$

corresponds to a gaussian distribution of dimension 2, one may interpret, for any fixed $x > y > 0$, $\varphi_0(x)$ as

$$\varphi_0(x) = \mathbb{P}\left(\ln\left(\alpha e^{Z_1} + (1-\alpha)e^{Z_2}\right) \geq x\right) \qquad (Z_1, Z_2) \sim \mathcal{N}(m, \Theta)\,, \tag{5.27}$$

with $\alpha = \alpha_1$. Θ may be re-expressed as

$$\Theta = \begin{pmatrix} v_1^2 & \theta \\ \theta & v_2^2 \end{pmatrix} \qquad \theta = \rho_J\, v_1 v_2. \tag{5.28}$$

$$
\begin{aligned}
\eta(0,x) &= \mathbb{P}\left(\ln\left(\alpha e^{Z_1}+(1-\alpha)e^{Z_2}\right)\geq x\right)\\
&= \mathbb{P}\left(Z_1\geq \ln\left(e^x-(1-\alpha)e^{Z_2}\right)-\ln(\alpha), Z_2\leq x-\ln(1-\alpha)\right)\\
&= \mathbb{P}\left(Z_1\geq \ln\left(e^x-(1-\alpha)e^{Z_2}\right)-\ln(\alpha)\middle| Z_2\leq x-\ln(1-\alpha)\right)\\
&\qquad\qquad\qquad\qquad \times \mathbb{P}\left(Z_2\leq x-\ln(1-\alpha)\right).
\end{aligned}
$$

Denote Φ the cumulative distribution function of a standard gaussian distribution that is

$$
\Phi(x)=\mathbb{P}(Z\leq x) \qquad Z\sim\mathcal{N}(0,1), \tag{5.29}
$$

and $g_{0,1}$ the distribution of a standard gaussian distribution:

$$
g_{0,1}(a)=\frac{1}{\sqrt{2\pi}}\exp\left(-\frac{1}{2}a^2\right). \tag{5.30}
$$

Since

$$
\mathcal{L}\left(Z_1|Z_2=a\right)=\mathcal{N}\left(m_1+\frac{\theta}{v_2^2}(a-m_2), v_1^2-\theta^2 v_2^2\right),
$$

then,

$$
\begin{aligned}
\varphi_0(x) = &\int_{-\infty}^{x-\ln(1-\alpha)} \mathbb{P}\left(Z_1\geq \ln\left(e^x-(1-\alpha)e^a\right)-\ln(\alpha)\middle| Z_2=a\right)\\
&\qquad\qquad \times\ g_{0,1}\left(\frac{a-m_2}{v_2}\right)\mathbb{P}\left(Z_2\leq x-\ln(1-\alpha)\right)\\
= &\int_{-\infty}^{x-\ln(1-\alpha)} \left[1-\Phi\left(\frac{\ln\left(e^x-(1-\alpha)e^a\right)-\ln(\alpha)-m_1-\frac{\theta}{v_2^2}(a-m_2)}{\sqrt{v_1^2-\theta^2v_2^2}}\right)\right]\\
&\qquad\qquad \times\ g_{0,1}\left(\frac{a-m_2}{v_2}\right)\times\Phi\left(\frac{x-\ln(1-\alpha)-m_2}{v_2}\right).
\end{aligned} \tag{5.31}
$$

Let us summarize the approximation method for the price of the index call option in the following algorithm,

Algorithm 5.1. *To approximate $C(T,K)$, follow this approximation method:*

1. *Compute in an explicit manner the values of $\varphi_0(x)$ via equation (5.31) on a certain grid $\mathcal{G}$ of $x\in[\ln\left(\frac{K}{I_0}\right),+\infty[$;*

2. *Compute* $\eta_0\left(\ln\left(\frac{K}{I_0}\right)\right)$ *by numerical quadrature using (5.8) on the grid* $\mathcal{G}$.

3. *Approximate* $C(T,K)$ *via*

$$C(T,K) \sim T\, I_0\, \eta_0\left(\ln\left(\frac{K}{I_0}\right)\right).$$

We now present a numerical example. Let us use the following parameters for the model (5.11):

$$S_0^1 = S_0^2 = 100, \quad r = 0.04, \quad \lambda = 2,$$

$$\Sigma^W = \begin{pmatrix} 1 & 0.1 \\ 0.1 & 1 \end{pmatrix}, \quad \Sigma^J = \begin{pmatrix} 1 & 0.4 \\ 0.4 & 1 \end{pmatrix},$$

$$\delta^1 = 0.1, \quad \delta^2 = 0.15, \quad v_1 = 0.1, \quad v_2 = 0.2, \quad m_1 = -0.05, \quad m_2 = -0.01.$$

1. We compute $\hat{C}_0(T,K)$, the price of the index call option for a maturity

$$T = \frac{20}{252}$$

and strikes K,

$$K = 101\ 102\ 103 \cdots\ 115\ 116\ 117,$$

using a Monte Carlo method with $N = 10^5$ trajectories. Denote $\hat{C}_0(T,K)$ this Monte Carlo estimator and $\hat{\sigma}(T,K)$ its standard deviation. A 90%-confidence interval for $C(T,K)$ is then given by

$$[LI_{90\%}(T,K); UI_{90\%}(T,K)]$$

$$= \left[\hat{C}_0(T,K) - \frac{1.96\hat{\sigma}(T,K)}{\sqrt{N}}\ ;\ \hat{C}_0(T,K) + \frac{1.96\hat{\sigma}(T,K)}{\sqrt{N}}\right]. \tag{5.32}$$

2. We follow the Algorithm 5.1 to approximate $C(T,K)$ by $Tl(K)$.

3. We compare the values $l(K)$ to $\hat{C}(T,K)/T$ and $LI_{10\%}(T,K)/T$, $UI_{10\%}(T,K)/T$, for different values of K. We summarize those results in the Figure 5.3.1 and Table 5.3.1.

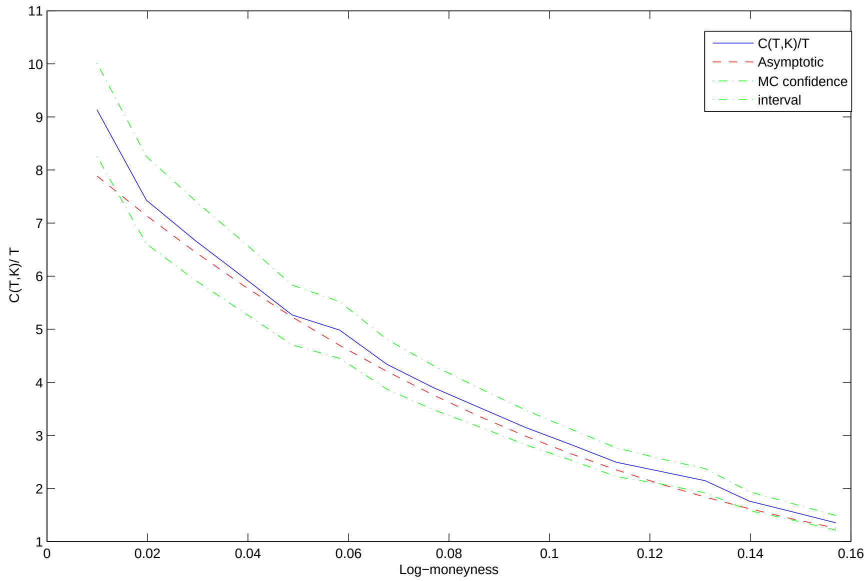

Figure 5.1: d=2 : Monte Carlo estimator $\hat{C}(T, K)$ vs analytical approximation (red) for 20-days call option prices.

5.3.2 The general case

We recall that η_0 is defined by,

$$\eta_0(y) = \int_y^\infty (e^u - e^y)\, k(0, u)\, du. \tag{5.33}$$

If one defines

$$g_{u,\epsilon}(z) = \frac{1}{\sqrt{2\pi}\epsilon} \exp\left(-\frac{1}{2\epsilon^2}\left(u - \ln\left(\sum_1^d \alpha_i e^{z_i}\right)\right)^2\right) \tag{5.34}$$

and

$$k_\epsilon(u) = \lambda \int_{\mathbb{R}^d} g_{u,\epsilon}(z)\, g_{m,\Theta}(z)\, dz, \tag{5.35}$$

K	$\hat{C}(T,K)/T$	$l(K)$	$LI_{10\%}(T,K)$	$UI_{10\%}(T,K)$
101	9.13	7.89	8.26	10.01
102	7.43	7.14	6.61	8.25
103	6.67	6.45	5.92	7.42
104	5.97	5.82	5.31	6.63
105	5.27	5.23	4.70	5.83
106	4.984	4.69	4.451	5.517
107	4.34	4.20	3.87	4.81
108	3.90	3.76	3.48	4.31
109	3.52	3.35	3.16	3.88
110	3.15	2.98	2.82	3.47
111	2.82	2.65	2.53	3.12
112	2.49	2.35	2.22	2.764
113	2.32	2.08	2.08	2.56
114	2.15	1.84	1.92	2.37
115	1.76	1.62	1.58	1.94
116	1.56	1.43	1.40	1.71
117	1.35	1.25	1.21	1.49

Table 5.1: d=2: Monte Carlo estimator $\hat{C}(T,K)$ vs analytical approximation (red) for 20-days call option prices $(LI_{10\%}(T,K), UI_{10\%}(T,K))$ is the 90% confidence interval for the Monte Carlo estimator.

then given equation (5.4), under Assumption 5.2, a dominated convergence argument yields

$$\lim_{\epsilon\to 0}\int_y^\infty (e^u - e^y)\, k_\epsilon(u)\, du = \eta_0(y). \tag{5.36}$$

Let us define $F_{u,\epsilon}$ by

$$F_{u,\epsilon}(z_1,\cdots,z_d) = \frac{1}{2\epsilon^2}\left(u - \ln\left(\sum_1^d \alpha_i e^{z_i}\right)\right)^2 + \frac{1}{2}{}^t(z-m)\Theta^{-1}(z-m), \tag{5.37}$$

then $k_\epsilon(u)$ rewrites

$$k_\epsilon(u) = \lambda\frac{1}{2\pi\epsilon}\,\frac{1}{(2\pi)^{d/2}\,|\Theta|^{1/2}}\int_{\mathbb{R}^d}\exp\left(-F_{u,\epsilon}(z_1,\cdots,z_d)\right)\, dz_1\,\cdots\, dz_d. \tag{5.38}$$

We propose a Laplace approximation to approximate the d-dimensional integral involved in equation (5.38), which consists essentially in observing that $\exp(-F_{u,\epsilon})$ is strongly peaked in a certain point of $\mathbb{R}^d$. Namely,

Proposition 5.1 (Laplace approximation)**.** *Assume that $F_{u,\epsilon}$ admits a global minimum on $\mathbb{R}^d$ at $z^\star_{u,\epsilon}$. Then*

$$k_\epsilon(u) \simeq \lambda \frac{1}{2\pi\epsilon} \frac{1}{\left(\left|\nabla^2 F_{u,\epsilon}(z^\star_{u,\epsilon})\right| \, |\Theta|\right)^{1/2}} \exp\left(-F_{u,\epsilon}(z^\star_{u,\epsilon})\right). \tag{5.39}$$

Proof. If $F_{u,\epsilon}$ admits a global minimum at $z^\star_{u,\epsilon}$, Taylor's formula for $F_{u,\epsilon}$ on the neighborhood of $z^\star_{u,\epsilon}$ yields

$$F_{u,\epsilon}(z) = F_{u,\epsilon}(z^\star_{u,\epsilon}) + \frac{1}{2}{}^t(z - z^\star_{u,\epsilon})\nabla^2 F_{u,\epsilon}(z^\star_{u,\epsilon})(z - z^\star_{u,\epsilon}) + \circ\left(\frac{\|z - z^\star_{u,\epsilon}\|^2}{2}\right),$$

leading to the following approximation,

$$\begin{aligned}
& k_\epsilon(u) \\
\sim \quad & \lambda \frac{1}{2\pi\epsilon} \frac{1}{(2\pi)^{d/2}\,|\Theta|^{1/2}} \exp\left(-F_{u,\epsilon}(z^\star_{u,\epsilon})\right) \int_{\mathbb{R}^d} \exp\left(-\frac{1}{2}{}^t(z - z^\star_{u,\epsilon})\nabla^2 F_{u,\epsilon}(z^\star_{u,\epsilon})(z - z^\star_{u,\epsilon})\right) \\
= \quad & \lambda \frac{1}{2\pi\epsilon} \frac{1}{(2\pi)^{d/2}\,|\Theta|^{1/2}} \exp\left(-F_{u,\epsilon}(z^\star_{u,\epsilon})\right) (2\pi)^{d/2} \left|\nabla^2 F_{u,\epsilon}(z^\star_{u,\epsilon})^{-1}\right|^{1/2} \\
= \quad & \lambda \frac{1}{2\pi\epsilon} \frac{1}{\left(\left|\nabla^2 F_{u,\epsilon}(z^\star_{u,\epsilon})\right| \, |\Theta|\right)^{1/2}} \exp\left(-F_{u,\epsilon}(z^\star_{u,\epsilon})\right).
\end{aligned}$$

□

In virtue of Proposition 5.1, for a given $u > 0$, if one wants to compute $k_\epsilon(u)$, then all the difficulty consists in locating the global minimum of $F_{u,\epsilon}$. To do so we solve the equation

$$\nabla F_{u,\epsilon}(z) = 0 \tag{5.40}$$

using a Newton-Raphson algorithm.

Let us summarize the approximation method of the index option:

Algorithm 5.2. *To approximate $C(T, K)$, follow this approximation method:*

1. *Choose a suitable value of ϵ: numerical experimentation shows that $\epsilon = 10^{-3}$ is enough;*

2. *Solve*
$$\nabla F_{u,\epsilon}(z^\star_{u,\epsilon}) = 0,$$
via the Newton-Raphson algorithm;

3. *Compute $k_\epsilon(u)$ via equation (5.39) on a certain grid $\mathcal{G}$ of the interval $\left[\ln\left(\frac{K}{I_0}\right);\infty\right]$;*

4. *Approximate $C(T,K)$ via*
$$C(T,K) \sim T\, I_0 \int_{\ln\left(\frac{K}{I_0}\right)}^{\infty} \left(e^u - \frac{K}{I_0}\right) k_\epsilon(u)\, du,$$
and compute this one-dimensional integral numerically on the grid $\mathcal{G}$.

We present two numerical examples:

1. Considering again the two-dimensional case, we intend to compare the values of $l(K)$ computed exactly via the probabilistic interpretation in Subsection 5.3.1 and the approximation obtained via the Laplace method. Our aim is, via numerical example, to check if the Laplace approximation is consistent enough. With the same parameters defined in Subsection 5.3.1, Table 1 represents the numerical computation of the asymptotics $l(K)$ computed via this two different methods. Results shows that the Laplace method is a good approximation leading to a mean absolute error of 0.4 %.

2. Let us now choose a basket of $d = 30$ assets, following the model (5.11) and specified by the parameters,
$$S_0^i = 100, \quad r = 0.04, \quad \lambda = 2, \quad \rho_W = 0.1, \quad \rho_J = 0.4\,. \tag{5.41}$$
We generate the δ^i, m_i, v_i's uniformly via
$$\delta^i \sim \mathcal{U}_{[0,1;0,2]}, \quad v_i \sim \mathcal{U}_{[0,1;0,2]}, \quad m_i \sim \mathcal{U}_{[-0,1;-0,01]}\,. \tag{5.42}$$
As in Subsection 5.3.1, for a maturity $T = 10/252$, we compute $\hat{C}(T,K)$ via Monte Carlo (See step 1 in Subsection 5.3.1).

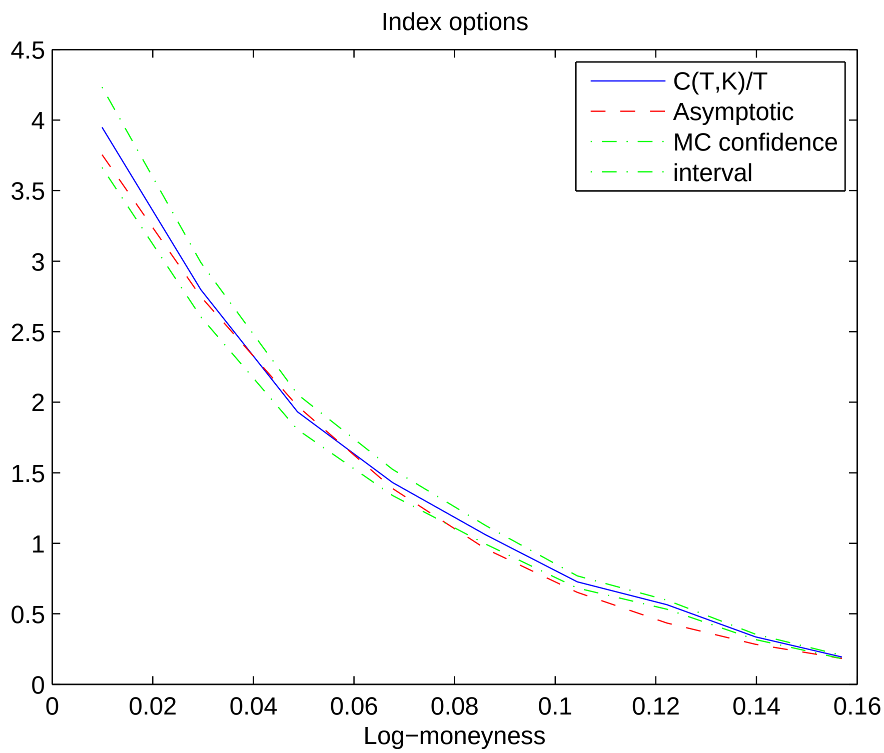

Figure 5.2: Dimension 30 : Monte Carlo estimator $\hat{C}(T, K)$ vs analytical approximation (red) for 20-days call option prices $(LI_{10\%}(T, K), UI_{10\%}(T, K))$ is the 90% confidence interval for the Monte Carlo estimator..

	Exact computation	Analytical-Laplace approximation
K	$l(K)$	$l(K)$
101	7.89	7.87
102	7.14	7.12
103	6.45	6.43
104	5.82	5.80
105	5.23	5.22
106	4.69	4.68
107	4.20	4.19
108	3.76	3.75
109	3.35	3.34
110	2.98	2.97
111	2.65	2.64
112	2.35	2.34
113	2.08	2.07
114	1.84	1.83
115	1.62	1.61
116	1.43	1.42
117	1.25	1.25

Table 5.2: Dimension 2 : Assessment of the accuracy of the Laplace approximation of the asymptotic $l(K)$.

We choose $\epsilon = 10^-3$ and follow the Algorithm 5.2 to approximate $C(T, K)$ by the asymptotic $l(K)$ at order T.

We summarize the results in Figure 5.2 and Table 5.3.2.

Remark 5.1. *In the particular case when the covariation matrix of the jumps Θ is homogeneous, then one can compute explicitly $z^\star_{u,\epsilon}$. Θ^{-1} is then of the form $\Theta^{-1} = (a_1 - a_2)\, I_d + a_1\, U$ where*

$$U = \begin{pmatrix} 1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1 \end{pmatrix}, \quad I_d = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix}.$$

Define

$$u^\star_\epsilon = \frac{1}{a_1 + (d-1)a_2}(a_1 - a_2)(1 + (a_1 + (d-1)a_2)d\epsilon^2) + m. \tag{5.43}$$

K	$\hat{C}(T,K)/T$	$l(K)$	$LI_{10\%}(T,K)$	$UI_{10\%}(T,K)$
101	3.95	3.75	3.66	4.23
103	2.80	2.75	2.61	3.00
105	1.93	1.97	1.81	2.06
107	1.43	1.39	1.34	1.52
109	1.06	0.96	0.99	1.13
111	0.73	0.65	0.68	0.77
113	0.56	0.44	0.53	0.60
115	0.34	0.28	0.32	0.36
117	0.19	0.18	0.18	0.20

Table 5.3: $d = 30$: Monte Carlo estimator $\hat{C}(T,K)$ vs analytical approximation (red) for 20-days call option prices $(LI_{10\%}(T,K), UI_{10\%}(T,K))$ is the 90% confidence interval for the Monte Carlo estimator.

Then for all u in $[0, u^\star_\epsilon[$, $F_{u,\epsilon}$ admits a global minimum at

$$z^\star_{u,\epsilon} = \left(m + \frac{u-m}{1+(a_1+(d-1)a_2)d\epsilon^2}, m + \frac{u-m}{1+(a_1+(d-1)a_2)d\epsilon^2} \right). \tag{5.44}$$

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